ADVERTISEMENT

The Smithsonian Institution has maintained for many years a group of publications in the nature of handy books of information on geographical, meteorological, physical, and mathematical subjects. These include the Smithsonian Geographical Tables (third edition, reprint, 1918); the Smithsonian Meteorological Tables (fourth revised edition, 1918); the Smithsonian Physical Tables (seventh revised edition, 1921); and the Smithsonian Mathematical Tables: Hyperbolic Functions (second reprint, 1921).

The present volume comprises the most important formulae of many branches of applied mathematics, an illustrated discussion of the methods of mechanical integration, and tables of elliptic functions. The volume has been compiled by Dr. E. P. Adams, of Princeton University. Prof. F. R. Moulton, of the University of Chicago, contributed the section on numerical solution of differential equations. The tables of elliptic functions were prepared by Col. R. L. Hippisley, C. B., under the direction of Sir George Greenhill, Bart., who has contributed the introduction to these tables.

The compiler, Dr. Adams, and the Smithsonian Institution are indebted to many physicists and mathematicians, especially to Dr. H. L. Curtis and colleagues of the Bureau of Standards, for advice, criticism, and coöperation in the preparation of this volume.

Charles D. Walcott,
Secretary of the Smithsonian Institution.

May, 1922.

PREFACE

The original object of this collection of mathematical formulae was to bring together, compactly, some of the more useful results of mathematical analysis for the benefit of those who regard mathematics as a tool, and not as an end in itself. There are many such results that are difficult to remember, for one who is not constantly using them, and to find them one is obliged to look through a number of books which may not immediately be accessible.

A collection of formulae, to meet the object of the present one, must be largely a matter of individual selection; for this reason this volume is issued in an interleaved edition, so that additions, meeting individual needs, may be made, and be readily available for reference.

It was not originally intended to include any tables of functions in this volume, but merely to give references to such tables. An exception was made, however, in favor of the tables of elliptic functions, calculated, on Sir George Greenhill's new plan, by Colonel Hippisley, which were fortunately secured for this volume, inasmuch as these tables are not otherwise available.

In order to keep the volume within reasonable bounds, no tables of indefinite and definite integrals have been included. For a brief collection, that of the late Professor B. O. Peirce can hardly be improved upon; and the elaborate collection of definite integrals by Bierens de Haan show how inadequate any brief tables of definite integrals would be. A short list of useful tables of this kind, as well as of other volumes, having an object similar to this one, is appended.

Should the plan of this collection meet with favor, it is hoped that suggestions for improving it and making it more generally useful may be received.

To Professor Moulton, for contributing the chapter on the Numerical Integration of Differential Equations, and to Sir George Greenhill, for his Introduction to the Tables of Elliptic Functions, I wish to express my gratitude. And I wish also to record my obligations to the Secretary of the Smithsonian Institution, and to Dr. C. G. Abbot, Assistant Secretary of the Institution, for the way in which they have met all my suggestions with regard to this volume.

E. P. ADAMS

PRINCETON, NEW JERSEY

COLLECTIONS OF MATHEMATICAL FORMULAE, ETC.

B. O. Peirce: A Short Table of Integrals, Boston, 1899.

G. Petit Bois: Tables d'Integrales Indefinies, Paris, 1906.

T. J. I'A. Bromwich: Elementary Integrals, Cambridge, 1911.

D. Bierens de Haan: Nouvelles Tables d'Integrales Definies, Leiden, 1867.

E. Jahnke and F. Emde: Funktionentafeln mit Formeln und Kurven, Leipzig, 1909.

G. S. Carr: A Synopsis of Elementary Results in Pure and Applied Mathematics, London, 1880.

W. Laska: Sammlung von Formeln der reinen und angewandten Mathematik, Braunschweig, 1888–1894.

W. Ligowski: Taschenbuch der Mathematik, Berlin, 1893.

O. Th. Bürklen: Formelsammlung und Repetitorium der Mathematik, Berlin, 1922.

F. Auerbach: Taschenbuch für Mathematiker und Physiker, 1. Jahrgang, 1909. Leipzig, 1909.

CONTENTS

SYMBOLS

log — logarithm. Whenever used the Naperian logarithm is understood. To find the common logarithm to base 10:

$$\log_{10} a = 0.43429 \ldots \log a.$$
$$\log a = 2.30259 \ldots \log_{10} a.$$

! — Factorial. $n!$ where n is an integer denotes $1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \ldots \cdot n$. Equivalent notation $\underline{n|}$

$\neq$ — Does not equal.

$>$ — Greater than.

$<$ — Less than.

$\geqslant$ — Greater than, or equal to.

$\leqslant$ — Less than, or equal to.

$\binom{n}{k}$ — Binomial coefficient. See 1.51.

$\rightarrow$ — Approaches.

$|a_{ik}|$ — Determinant where a_{ik} is the element in the ith row and kth column.

$\dfrac{\partial(u_1, u_2, \ldots)}{\partial(x_1, x_2, \ldots)}$ — Functional determinant. See 1.37.

$|a|$ — Absolute value of a. If a is a real quantity its numerical value, without regard to sign. If a is a complex quantity, $a = \alpha + i\beta$, $|a| =$ modulus of $a = +\sqrt{\alpha^2 + \beta^2}$.

i — The imaginary $= +\sqrt{-1}$.

$\sum$ — Sign of summation, i.e., $\sum_{k=1}^{k=n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$.

$\prod$ — Product, i.e., $\prod_{k=1}^{k=n} (1 + kx) = (1 + x)(1 + 2x)(1 + 3x) \ldots (1 + nx)$.

I. ALGEBRA

1.00 Algebraic Identities.

1. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots\cdots + ab^{n-2} + b^{n-1})$.

2. $a^n \pm b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots\cdots \mp ab^{n-2} \pm b^{n-1})$.

   n odd: upper sign.
   n even: lower sign.

3. $(x + a_1)(x + a_2) \cdots\cdots (x + a_n) = x^n + P_1x^{n-1} + P_2x^{n-2} + \cdots\cdots + P_{n-1}x + P_n.$

   $P_1 = a_1 + a_2 + \cdots\cdots\cdots + a_n.$
   $P_k =$ sum of all the products of the a's taken k at a time.
   $P_n = a_1a_2a_3 \cdots\cdots a_n.$

4. $(a^2 + b^2)(\alpha^2 + \beta^2) = (a\alpha \mp b\beta)^2 + (a\beta \pm b\alpha)^2$.

5. $(a^2 - b^2)(\alpha^2 - \beta^2) = (a\alpha \pm b\beta)^2 - (a\beta \pm b\alpha)^2$.

6. $(a^2 + b^2 + c^2)(\alpha^2 + \beta^2 + \gamma^2) = (a\alpha + b\beta + c\gamma)^2 + (b\gamma - \beta c)^2 + (c\alpha - \gamma a)^2 + (a\beta - \alpha b)^2$.

7. $(a^2 + b^2 + c^2 + d^2)(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) = (a\alpha + b\beta + c\gamma + d\delta)^2 + (a\beta - b\alpha + c\delta - d\gamma)^2 + (a\gamma - b\delta - c\alpha + d\beta)^2 + (a\delta + b\gamma - c\beta - d\alpha)^2$.

8. $(ac - bd)^2 + (ad + bc)^2 = (ac + bd)^2 + (ad - bc)^2$.

9. $(a + b)(b + c)(c + a) = (a + b + c)(ab + bc + ca) - abc$.

10. $(a + b)(b + c)(c + a) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 2abc$.

11. $(a + b)(b + c)(c + a) = bc(b + c) + ca(c + a) + ab(a + b) + 2abc$.

12. $3(a + b)(b + c)(c + a) = (a + b + c)^3 - (a^3 + b^3 + c^3)$.

13. $(b - a)(c - a)(c - b) = a^2(c - b) + b^2(a - c) + c^2(b - a)$.

14. $(b - a)(c - a)(c - b) = a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

15. $(b - a)(c - a)(c - b) = bc(c - b) + ca(a - c) + ab(b - a)$.

16. $(a - b)^2 + (b - c)^2 + (c - a)^2 = 2[(a - b)(a - c) + (b - a)(b - c) + (c - a)(c - b)]$.

17. $a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2) = (a - b)(b - c)(a - c)(ab + bc + ca)$.

18. $(a + b + c)(a^2 + b^2 + c^2) = bc(b + c) + ca(c + a) + ab(a + b) + a^3 + b^3 + c^3$.

19. $(a + b + c)(bc + ca + ab) = a^2(b + c) + b^2(c + a) + c^2(a + b) + 3abc$.

20. $(b + c - a)(c + a - b)(a + b - c) = a^2(b + c) + b^2(c + a) + c^2(a + b)$

21. $(a+b+c)(-a+b+c)(a-b+c)(a+b-c) = 2(b^2c^2+c^2a^2+a^2b^2)$
$\quad -(a^4+b^4+c^4).$

22. $(a+b+c+d)^2+(a+b-c-d)^2+(a+c-b-d)^2+(a+d-b-c)^2$
$\quad = 4(a^2+b^2+c^2+d^2).$

If
$$A = a\alpha + b\gamma + c\beta$$
$$B = a\beta + b\alpha + c\gamma$$
$$C = a\gamma + b\beta + c\alpha$$

23. $(a+b+c)(\alpha+\beta+\gamma) = A+B+C.$

24. $[a^2+b^2+c^2-(ab+bc+ca)][\alpha^2+\beta^2+\gamma^2-(\alpha\beta+\beta\gamma+\gamma\alpha)]$
$\quad = A^2+B^2+C^2-(AB+BC+CA).$

25. $(a^3+b^3+c^3-3abc)(\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma) = A^3+B^3+C^3-3ABC.$

ALGEBRAIC EQUATIONS

1.200 The expression

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots\dots + a_{n-1}x + a_n$$

is an integral rational function, or a polynomial, of the nth degree in

1.201 The equation $f(x) = 0$ has n roots which may be real or comple tinct or repeated.

1.202 If the roots of the equation $f(x) = 0$ are $c_1, c_2, \dots, c_n$,

$$f(x) = a_0(x-c_1)(x-c_2)\dots\dots(x-c_n)$$

1.203 Symmetric functions of the roots are expressions giving certain binations of the roots in terms of the coefficients. Among the more imp are:

$$c_1 + c_2 + \dots\dots\dots\dots + c_n = -\frac{a_1}{a_0}$$

$$c_1c_2 + c_1c_3 + \dots + c_2c_3 + c_2c_4 + \dots\dots\dots + c_{n-1}c_n = \frac{a_2}{a_0}$$

$$c_1c_2c_3 + c_1c_2c_4 + \dots + c_1c_3c_4 + \dots\dots + c_{n-2}c_{n-1}c_n = -\frac{a_3}{a_0}$$

$$\dots\dots\dots\dots$$

$$\dots\dots\dots\dots$$

$$c_1c_2c_3\dots\dots c_n = (-1)^n\frac{a_n}{a_0}.$$

1.204 Newton's Theorem. If s_k denotes the sum of the kth powers of roots of $f(x) = 0$,

$$s_k = c_1^k + c_2^k + \dots\dots\dots + c_n^k$$

$$1a_1 + s_1a_0 = 0$$

$$2a_2 + s_1a_1 + s_2a_0 = 0$$

$$3a_3 + s_1a_2 + s_2a_1 + s_3a_0 = 0$$

$$4a_4 + s_1a_3 + s_2a_2 + s_3a_1 + s_4a_0 = 0$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

or:

$$s_1 = -\frac{a_1}{a_0}$$

$$s_2 = -\frac{2a_2}{a_0} + \frac{a_1^2}{a_0^2}$$

$$s_3 = -\frac{3a_3}{a_0} + \frac{3a_1a_2}{a_0^2} - \frac{a_1^3}{a_0^3}$$

$$s_4 = -\frac{4a_4}{a_0} + \frac{4a_1a_3}{a_0^2} - \frac{4a_1^2a_2}{a_0^3} + \frac{2a_2^2}{a_0^2} + \frac{a_1^4}{a_0^4}$$

. . . .

. . . .

1.205 If S_k denotes the sum of the reciprocals of the kth powers of all the roots of the equation $f(x) = 0$:

$$S_k = \frac{1}{c_1^k} + \frac{1}{c_2^k} + \dots\dots + \frac{1}{c_n^k}$$

$$1a_{n-1} + S_1a_n = 0$$

$$2a_{n-2} + S_1a_{n-1} + S_2a_n = 0$$

$$3a_{n-3} + S_1a_{n-2} + S_2a_{n-1} + S_3a_n = 0$$

. . . . . .

. . . . . .

$$S_1 = -\frac{a_{n-1}}{a_n}$$

$$S_2 = -\frac{2a_{n-2}}{a_n} + \frac{a^2_{n-1}}{a_n^2}$$

$$S_3 = -\frac{3a_{n-3}}{a_n} + \frac{3a_{n-1}a_{n-2}}{a_n^2} - \frac{a^3_{n-1}}{a^3_n}$$

. . . .

. . . .

. . . .

1.220 If $f(x)$ is divided by $x - h$ the result is

$$f(x) = (x - h)Q + R.$$

Q is the quotient and R the remainder. This operation may be readily performed as follows:

Write in line the values of $a_0, a_1, \dots\dots, a_n$. If any power of x is missing write 0 in the corresponding place. Multiply a_0 by h and place the product in the second line under a_1; add to a_1 and place the sum in the third line under a_1. Multiply this sum by h and place the product in the second line under a_2; add to a_2 and place the sum in the third line under a_2. Continue this series of operations until the third line is full. The last term in the third line is the remainder, R. The first term in the third line, which is a_0, is the coefficient of x^{n-1} in the quotient, Q; the second term is the coefficient of x^{n-2}, and so on.

1.221 It follows from **1.220** that $f(h) = R$. This gives a convenient way of evaluating $f(x)$ for $x = h$.

1.222 To express $f(x)$ in the form:

$$f(x) = A_0(x-h)^n + A_1(x-h)^{n-1} + \ldots\ldots + A_{n-1}(x-h) + A_n.$$

By **1.220** form $f(h) = A_n$. Repeat this process with each quotient, and the last term of each line of sums will be a succeeding value of the series of coefficients $A_n, A_{n-1}, \ldots\ldots, A_0$.

Example:

$$f(x) = 3x^5 + 2x^4 - 8x^2 + 2x - 4 \qquad h = 2$$

3	2	0	−8	2	−4
	6	16	32	48	100
3	8	16	24	50	96 = A_5
	6	28	88	224	
	14	44	112	274 = A_4	
	6	40	168		
	20	84	280 = A_3		
	6	52			
	26	136 = A_2			
	6				
	32 = A_1				
3 = A_0					

Thus:

$$Q = 3x^4 + 8x^3 + 16x^2 + 24x + 50$$
$$R = f(2) = 96$$
$$f(x) = 3(x-2)^5 + 32(x-2)^4 + 136(x-2)^3 + 280(x-2)^2 + 274(x-2) + 96$$

TRANSFORMATION OF EQUATIONS

1.230 To transform the equation $f(x) = 0$ into one whose roots all have their signs changed: Substitute $-x$ for x.

1.231 To transform the equation $f(x) = 0$ into one whose roots are all multiplied by a constant, m: Substitute x/m for x.

1.232 To transform the equation $f(x) = 0$ into one whose roots are the reciprocals of the roots of the given equation: Substitute $1/x$ for x and multiply by x^n.

1.233 To transform the equation $f(x) = 0$ into one whose roots are all increased or diminished by a constant, h: Substitute $x \pm h$ for x in the given equation.

the upper sign being used if the roots are to be diminished and the lower sign if they are to be increased. The resulting equation will be:

$$f(\pm h) + xf'(\pm h) + \frac{x^2}{2!}f''(\pm h) + \frac{x^3}{3!}f'''(\pm h) + \ldots\ldots = 0$$

where $f'(x)$ is the first derivative of $f(x)$, $f''(x)$, the second derivative, etc. The resulting equation may also be written:

$$A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \ldots\ldots\ldots + A_{n-1}x + A_n = 0$$

where the coefficients may be found by the method of **1.222** if the roots are to be diminished. To increase the roots by h change the sign of h.

MULTIPLE ROOTS

1.240 If c is a multiple root of $f(x) = 0$, of order m, i.e., repeated m times, then

$$f(x) = (x - c)^m Q; \qquad R = 0$$

c is also a multiple root of order $m - 1$ of the first derived equation, $f'(x) = 0$; of order $m - 2$ of the second derived equation, $f''(x) = 0$, and so on.

1.241 The equation $f(x) = 0$ will have no multiple roots if $f(x)$ and $f'(x)$ have no common divisor. If $F(x)$ is the greatest common divisor of $f(x)$ and $f'(x)$, $f(x)/F(x) = f_1(x)$, and $f_1(x)$ will have no multiple roots.

1.250 An equation of odd degree, n, has at least one real root whose sign is opposite to that of a_n.

1.251 An equation of even degree, n, has one positive and one negative real root if a_n is negative.

1.252 The equation $f(x) = 0$ has as many real roots between $x = x_1$ and $x = x_2$ as there are changes of sign in $f(x)$ between x_1 and x_2.

1.253 Descartes' Rule of Signs: No equation can have more positive roots than it has changes of sign from + to − and from − to +, in the terms of $f(x)$. No equation can have more negative roots than there are changes of sign in $f(-x)$.

1.254 If $f(x) = 0$ is put in the form

$$A_0(x - h)^n + A_1(x - h)^{n-1} + \ldots\ldots\ldots + A_n = 0$$

by **1.222**, and $A_0, A_1, \ldots\ldots, A_n$ are all positive, h is an upper limit of the positive roots.

If $f(1/x) = 0$ is put in a similar form, and the coefficients are all positive, h is a lower limit of the positive roots. And with $f(-1/x) = 0$, h is an upper limit of the negative roots.

1.255 Sturm's Theorem. Form the functions:

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n$$
$$f_1(x) = f'(x) = na_0x^{n-1} + (n-1)a_1x^{n-2} + \ldots\ldots + a_{n-1}$$
$$f_2(x) = -R_1 \text{ in } f(x) = Q_1f_1(x) + R_1$$
$$f_3(x) = -R_2 \text{ in } f_1(x) = Q_2f_2(x) + R_2$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$

The number of real roots of $f(x) = 0$ between $x = x_1$ and $x = x_2$ is equal to the number of changes of sign in the series $f(x)$, $f_1(x)$, $f_2(x)$, . . . when x_1 is substituted for x minus the number of changes of sign in the same series when x_2 is substituted for x. In forming the functions f_1, f_2, ... numerical factors may be introduced or suppressed in order to remove fractional coefficients.

Example:

$$f(x) = x^4 - 2x^3 - 3x^2 + 10x - 4$$
$$f_1(x) = 2x^3 - 3x^2 - 3x + 5$$
$$f_2(x) = 9x^2 - 27x + 11$$
$$f_3(x) = -8x - 3$$
$$f_4(x) = -1433$$

	f	f_1	f_2	f_3	f_4	
$x = -\infty$	+	−	+	+	−	3 changes
$x = 0$	−	+	+	−	−	2 changes
$x = +\infty$	+	+	+	−	−	1 change

Therefore there is one positive and one negative real root.

If it can be seen that all the roots of any one of Sturm's functions are imaginary it is unnecessary to calculate any more of them after that one.

If there are any multiple roots of the equation $f(x) = 0$ the series of Sturm's functions will terminate with f_r, $r < n$. $f_r(x)$ is the highest common factor of f and f_1. In this case the number of real roots of $f(x) = 0$ lying between $x = x_1$ and $x = x_2$, each multiple root counting only once, will be the difference between the number of changes of sign in the series $f, f_1, f_2, \ldots, f_r$ when x_1 and x_2 are successively substituted in them.

1.256 Routh's rule for finding the number of roots whose real parts are positive. (Rigid Dynamics, Part II, Art. 297.)

Arrange the coefficients in two rows:

x^n	a_0	a_2	a_4	. . . .
x^{n-1}	a_1	a_3	a_5	. . . .

Form a third row by cross-multiplication:

$$\ldots{-2} \qquad \frac{a_1a_2 - a_0a_3}{a_1} \qquad \frac{a_1a_4 - a_0a_5}{a_1} \qquad \frac{a_1a_6 - a_0a_7}{a_1} \qquad \ldots$$

orm a fourth row by operating on these last two rows by a similar cross-ultiplication. Continue this operation until there are no terms left. The umber of variations of sign in the first column gives the number of roots hose real parts are positive.

If there are any equal roots some of the subsidiary functions will vanish. n place of one which vanishes write the differential coefficient of the last one hich does not vanish and proceed in the same way. At the left of each row written the power of x corresponding to the first subsidiary function in that w. This power diminishes by 2 for each succeeding coefficient in the row.

Any row may be multiplied or divided by any positive quantity in order remove fractions.

DETERMINATION OF THE ROOTS OF AN EQUATION

260 Newton's Method. If a root of the equation $f(x) = 0$ is known to lie tween x_1 and x_2 its value can be found to any desired degree of approximation Newton's method. This method can be applied to transcendental equations well as to algebraic equations.

If b is an approximate value of a root,

$$b - \frac{f(b)}{f'(b)} = c \text{ is a second approximation,}$$

$$c - \frac{f(c)}{f'(c)} = d \text{ is a third approximation.}$$

his process may be repeated indefinitely.

261 Horner's Method for approximating to the real roots of $f(x) = 0$.

Let p_1 be the first approximation, such that $p_1 + 1 > c > p_1$, where c is the ot sought. The equation can always be transformed into one in which this ndition holds by multiplying or dividing the roots by some power of 10 **1.231.** Diminish the roots by p_1 by **1.233.** In the transformed equation

$$A_0(x - p_1)^n + A_1(x - p_1)^{n-1} + \ldots + A_{n-1}(x - p_1) + A_n = 0$$

t

$$\frac{p_2}{10} = \frac{A_n}{A_{n-1}}$$

d diminish the roots by $p_2/10$, yielding a second transformed equation

$$B_0\left(x - p_1 - \frac{p_2}{10}\right)^n + B_1\left(x - p_1 - \frac{p_2}{10}\right)^{n-1} + \ldots + B_n = 0.$$

If B_n and B_{n-1} are of the same sign p_3 was taken too large and must be diminished. Then take

$$\frac{p_3}{100} = \frac{B_n}{B_{n-1}}$$

and continue the operation. The required root will be:

$$c = p_1 + \frac{p_2}{10} + \frac{p_3}{100} + \ldots\ldots$$

1.262 Graeffe's Method. This method determines approximate values of all the roots of a numerical equation, complex as well as real. Write the equation of the nth degree

$$f(x) = a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \ldots\ldots \pm a_n = 0.$$

The product

$$f(x)\cdot f(-x) = A_0x^{2n} - A_1x^{2n-2} + A_2x^{2n-4} - \ldots\ldots \pm A_n = 0$$

contains only even powers of x. It is an equation of the nth degree in x^2. The coefficients are determined by.

$$\begin{aligned}
A_0 &= a_0^2 \\
A_1 &= a_1^2 - 2a_0a_2 \\
A_2 &= a_2^2 - 2a_1a_3 + 2a_0a_4 \\
A_3 &= a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_0a_6 \\
A_4 &= a_4^2 - 2a_3a_5 + 2a_2a_6 - 2a_1a_7 + 2a_0a_8 \\
&\ldots\ldots \\
&\ldots\ldots
\end{aligned}$$

The roots of the equation

$$A_0y^n - A_1y^{n-1} + A_2y^{n-2} - \ldots\ldots \pm A_n = 0$$

are the squares of the roots of the given equation. Continuing this process we get an equation

$$R_0u^n - R_1u^{n-1} + R_2u^{n-2} - \ldots\ldots \pm R_n = 0$$

whose roots are the 2^rth powers of the roots of the given equation. Put $\lambda = 2^r$. Let the roots of the given equation be $c_1, c_2, \ldots\ldots, c_n$. Suppose first that

$$c_1 > c_2 > c_3 > \ldots\ldots > c_n$$

Then for large values of λ,

$$c_1{}^\lambda = \frac{R_1}{R_0}, \qquad c_2{}^\lambda = \frac{R_2}{R_1}, \qquad \ldots\ldots, \qquad c_n{}^\lambda = \frac{R_n}{R_{n-1}}.$$

If the roots are real they may be determined by extracting the λth roots of these quantities. Whether they are $\pm$ is determined by taking the sign which approximately satisfies the equation $f(x) = 0$.

Suppose next that complex roots enter so that there are equalities among the absolute values of the roots. Suppose that

$$|c_1| \geqq |c_2| \geqq |c_3| \geqq \ldots\ldots \geqq |c_p|; \qquad |c_p| > |c_{p+1}|;$$
$$|c_{p+1}| \geqq |c_{p+2}| \geqq \ldots\ldots \geqq |c_n|$$

Then if λ is large enough so that $c_p{}^\lambda$ is large compared to $c_{p+1}{}^\lambda$, $c_1{}^\lambda$, $c_2{}^\lambda$, ... $c_p{}^\lambda$ approximately satisfy the equation:

$$R_0u^p - R_1u^{p-1} + R_2u^{p-2} - \ldots\ldots \pm R_p = 0$$

and $c_{p+1}{}^\lambda$, $c_{p+2}{}^\lambda$, ... , $c_n{}^\lambda$ approximately satisfy the equation:

$$R_pu^{n-p} - R_{p+1}u^{n-p-1} + R_{p+2}u^{n-p-2} - \ldots\ldots \pm R_n = 0.$$

Therefore when λ is large enough the given equation breaks down into a number of simpler equations. This stage is shown in the process of deriving the successive equations when certain of the coefficients are obtained from those of the preceding equation simply by squaring.

REFERENCES: Encyklopadie der Math. Wiss. I, 1, 3a (Runge). BAIRSTOW: Applied Aerodynamics, pp. 553–560; the solution of a numerical equation of the 8th degree is given by Graeffe's Method.

1.270 Quadratic Equations.

$$x^2 + 2ax + b = 0.$$

The roots are:

$$x_1 = -a + \sqrt{a^2 - b}$$
$$x_2 = -a - \sqrt{a^2 - b}$$
$$x_1 + x_2 = -2a$$
$$x_1x_2 = b.$$

If $a^2 > b$ roots are real,

$a^2 < b$ roots are complex,

$a^2 = b$ roots are equal.

1.271 Cubic equations.

$$(1)\quad x^3 + ax^2 + bx + c = 0.$$

Substitute

$$(2)\quad x = y - \frac{a}{3}$$

$$(3)\quad y^3 - 3py - 2q = 0$$

where

$$3p = \frac{a^2}{3} - b$$

$$2q = \frac{ab}{3} - \frac{2}{27}a^3 - c.$$

Roots of (3):

If $p > 0$, $q > 0$, $q^2 > p^3$

$$\cosh\phi = \frac{q}{\sqrt{p^3}}$$

$$y_1 = 2\sqrt{p}\ \cosh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{3p}\ \sinh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{3p}\ \sinh\frac{\phi}{3}.$$

If $p > 0$, $q < 0$, $q^2 > p^3$,

$$\cosh\phi = \frac{-q}{\sqrt{p^3}}$$

$$y_1 = -2\sqrt{p}\ \cosh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{3p}\ \sinh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{3p}\ \sinh\frac{\phi}{3}.$$

If $p < 0$

$$\sinh\phi = \frac{q}{\sqrt{-p^3}}$$

$$y_1 = 2\sqrt{-p}\ \sinh\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + i\sqrt{-3p}\ \cosh\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - i\sqrt{-3p}\ \cosh\frac{\phi}{3}.$$

If $p > 0$, $q^2 < p^3$,

$$\cos\phi = \frac{q}{\sqrt{p^3}}$$

$$y_1 = 2\sqrt{p}\ \cos\frac{\phi}{3}$$

$$y_2 = -\frac{y_1}{2} + \sqrt{3p}\ \sin\frac{\phi}{3}$$

$$y_3 = -\frac{y_1}{2} - \sqrt{3p}\ \sin\frac{\phi}{3}.$$

1.272 Biquadratic equations.

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0.$$

Substitute

$$x = y - \frac{a_1}{a_0}$$

$$y^4 + \frac{6}{a_0^2}Hy^2 + \frac{4}{a_0^3}Gy + \frac{1}{a_0^4}F = 0$$

$$H = a_0a_2 - a_1^2$$
$$G = a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3$$
$$F = a_0^3a_4 - 4a_0^2a_1a_3 + 6a_0a_1^2a_2 - 3a_1^4$$
$$I = a_0a_4 - 4a_1a_3 + 3a_2^2$$
$$F = a_0^2I - 3H^2$$
$$J = a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3$$
$$\Delta = I^3 - 27J^2 = \text{the discriminant}$$
$$G^2 + 4H^3 = a_0^2(HI - a_0J).$$

Nature of the roots of the biquadratic:

$\Delta = 0$ Equal roots are present
- Two roots only equal: I and J are not both zero
- Three roots are equal: $I = J = 0$
- Two distinct pairs of equal roots: $G = 0$; $a_0^2I - 12H^2 = 0$
- Four roots equal: $H = I = J = 0$.

$\Delta < 0$ Two real and two complex roots

$\Delta > 0$ Roots are either all real or all complex:
- $H < 0$ and $a_0^2I - 12H^2 < 0$ Roots all real
- $H > 0$ and $a_0^2I - 12H^2 > 0$ Roots all complex.

DETERMINANTS

1.300 A determinant of the nth order, with n^2 elements, is written:

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{vmatrix} = |a_{ij}|, \quad (i, j = 1, 2, \ldots, n)$$

1.301 A determinant is not changed in value by writing rows for columns and columns for rows.

1.302 If two columns or two rows of a determinant are interchanged the resulting determinant is unchanged in value but is of the opposite sign.

1.303 A determinant vanishes if it has two equal columns or two equal rows.

1.304 If each element of a row or a column is multiplied by the same factor

1.305 A determinant is not changed in value if to each element of a row or column is added the corresponding element of another row or column multiplied by a common factor.

1.306 If each element of the lth row or column consists of the sum of two or more terms the determinant splits up into the sum of two or more determinants having for elements of the lth row or column the separate terms of the lth row or column of the given determinant.

1.307 If corresponding elements of two rows or columns of a determinant have a constant ratio the determinant vanishes.

1.308 If the ratio of the differences of corresponding elements in the pth and qth rows or columns to the differences of corresponding elements in the rth and sth rows or columns be constant the determinant vanishes.

1.309 If p rows or columns of a determinant whose elements are rational integral functions of x become equal or proportional when $x = h$, the determinant is divisible by $(x-h)^{p-1}$.

MULTIPLICATION OF DETERMINANTS

1.320 Two determinants of equal order may be multiplied together by the scheme:

$$|a_{ij}| \times |b_{ij}| = |c_{ij}|$$

where

$$c_{ij} = a_{i1}b_{j1} + a_{i2}b_{j2} + \dots\dots + a_{in}b_{jn}.$$

1.321 If the two determinants to be multiplied are of unequal order the one of lower order can be raised to one of equal order by bordering it; i.e.:

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{11} & a_{12} & \dots & a_{1n} \\ 0 & 0 & 0 & \dots & a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

1.322 The product of two determinants may be written:

$$= \begin{vmatrix} a_{11} & \dots & a_{1n} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} & 0 & \dots & 0 \\ 0 & \dots & 0 & b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & b_{n1} & \dots & b_{nn} \end{vmatrix}$$

DIFFERENTIATION OF DETERMINANTS

1.330 If the elements of a determinant, Δ, are functions of a variable, t:

$$\frac{d\Delta}{dt} = \begin{vmatrix} a'_{11} & a_{12} & \dots & a_{1n} \\ a'_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a'_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a'_{12} & \dots & a_{1n} \\ a_{21} & a'_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a'_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$+ \dots\dots\dots\dots + \begin{vmatrix} a_{11} & a_{12} & \dots & a'_{1n} \\ a_{21} & a_{22} & \dots & a'_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a'_{nn} \end{vmatrix}$$

where the accents denote differentiation by t.

EXPANSION OF DETERMINANTS

1.340 The complete expansion of a determinant of the nth order contains $n!$ terms. Each of these terms contains one element from each row and one element from each column. Any term may be obtained from the leading term:

$$a_{11}a_{22}a_{33} \dots\dots\dots a_{nn}$$

by keeping the first suffixes unchanged and permuting the second suffixes among $1, 2, 3, \dots, n$. The sign of any term is determined by the number of inversions from the second suffixes of the leading term, being positive if there is an even number of inversions and negative if there is an odd number of inversions.

1.341 The coefficient of a_{ij} when the determinant Δ is fully expanded is:

Δ_{ij} is the first minor of the determinant Δ corresponding to a_{ij} and is a determinant of order $n-1$. It may be obtained from Δ by crossing out the row and column which intersect in a_{ij} and multiplying by $(-1)^{i+j}$.

1.342

$$a_{i1}\Delta_{j1}+a_{i2}\Delta_{j2}+\cdots\cdots+a_{in}\Delta_{jn}=\begin{matrix}0 & \text{if } i\neq j\\ \Delta & \text{if } i=j\end{matrix}$$

$$a_{1i}\Delta_{1j}+a_{2i}\Delta_{2j}+\cdots\cdots+a_{ni}\Delta_{nj}=\begin{matrix}0 & \text{if } i\neq j\\ \Delta & \text{if } i=j\end{matrix}.$$

1.343

$$\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}}=\frac{\partial\Delta_{kl}}{\partial a_{ij}}=\frac{\partial\Delta_{ij}}{\partial a_{kl}}$$

is the coefficient of $a_{ij}a_{kl}$ in the complete expansion of the determinant Δ. It may be obtained from Δ, except for sign, by crossing out the rows and columns which intersect in a_{ij} and a_{kl}.

1.344

$$|\Delta_{ij}|\times|a_{ij}|=\Delta^n$$
$$|\Delta_{ij}|=\Delta^{n-1}.$$

The determinant $|\Delta_{ij}|$ is the reciprocal determinant to Δ.

1.345

$$\Delta\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}}=\begin{vmatrix}\Delta_{ij} & \Delta_{il}\\ \Delta_{kj} & \Delta_{kl}\end{vmatrix}=\frac{\partial\Delta}{\partial a_{ij}}\frac{\partial\Delta}{\partial a_{kl}}-\frac{\partial\Delta}{\partial a_{il}}\frac{\partial\Delta}{\partial a_{kj}}.$$

1.346

$$\Delta^2\frac{\partial^3\Delta}{\partial a_{ij}\partial a_{kl}\partial a_{pq}}=\begin{vmatrix}\Delta_{ij} & \Delta_{il} & \Delta_{iq}\\ \Delta_{kj} & \Delta_{kl} & \Delta_{kq}\\ \Delta_{pj} & \Delta_{pl} & \Delta_{pq}\end{vmatrix}$$

1.347

$$\frac{\partial^2\Delta}{\partial a_{ij}\partial a_{kl}}=-\frac{\partial^2\Delta}{\partial a_{il}\partial a_{kj}}.$$

1.348 If $\Delta=0$,

$$\frac{\partial\Delta}{\partial a_{ij}}\frac{\partial\Delta}{\partial a_{kl}}=\frac{\partial\Delta}{\partial a_{il}}\frac{\partial\Delta}{\partial a_{kj}}.$$

1.350 If $a_{ij}=a_{ji}$ the determinant is symmetrical. In a symmetrical determinant

$$\Delta_{ij}=\Delta_{ji}.$$

1.351 If $a_{ij}=-a_{ji}$ the determinant is a skew determinant. In a skew determinant

1.352 If $a_{ij} = -a_{ji}$, and $a_{ii} = 0$, the determinant is a skew symmetrical determinant.

A skew symmetrical determinant of even order is a perfect square.

A skew symmetrical determinant of odd order vanishes.

1.360 A system of linear equations:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = k_1\\
&a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = k_2\\
&\dots\dots\dots\dots\\
&\dots\dots\dots\dots\\
&a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = k_n
\end{aligned}$$

has a solution:

$$\Delta \cdot x_i = k_1\Delta_{1i} + k_2\Delta_{2i} + \dots\dots + k_n\Delta_{ni}$$

provided that

$$\Delta = | a_{ij} | \neq 0.$$

1.361 If $\Delta = 0$, but all the first minors are not 0,

$$\Delta_{ss} \cdot x_j = x_s\Delta_{sj} + \sum_{r=1}^{n} k_r \frac{\partial^2 \Delta}{\partial a_{ss}\partial a_{rj}} \qquad (j = 1, 2, \dots n)$$

where s may be any one of the integers 1, 2, ..., n.

1.362 If $k_1 = k_2 = \dots\dots = k_n = 0$, the linear equations are homogeneous, and if $\Delta = 0$,

$$\frac{x_j}{\Delta_{sj}} = \frac{x_s}{\Delta_{ss}} \qquad (j = 1, 2, \dots n).$$

1.363 The condition that n linear homogeneous equations in n variables shall be consistent is that the determinant, Δ, shall vanish.

1.364 If there are $n + 1$ linear equations in n variables:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = k_1\\
&a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = k_2\\
&\dots\dots\dots\dots\dots\dots\\
&\dots\dots\dots\dots\dots\dots\\
&a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = k_n\\
&c_1x_1 + c_2x_2 + \dots\dots + c_nx_n = k_{n+1}
\end{aligned}$$

the condition that this system shall be consistent is that the determinant:

$$\begin{vmatrix}
a_{11} & a_{12} & \dots\dots & a_{1n} & k_1\\
a_{21} & a_{22} & \dots\dots & a_{2n} & k_2\\
\dots & \dots & \dots\dots & \dots & \dots\\
\dots & \dots & \dots\dots & \dots & \dots\\
a_{n1} & a_{n2} & \dots\dots & a_{nn} & k_n\\
c_1 & c_2 & \dots\dots & c_n & k_{n+1}
\end{vmatrix} = 0$$

1.370 Functional Determinants.

If $y_1, y_2, \ldots, y_n$ are n functions of $x_1, x_2, \ldots\ldots, x_n$:

$$y_k = f_k(x_1, x_2, \ldots\ldots, x_n)$$

the determinant:

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots\cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots\cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots\cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} = \left|\frac{\partial y_i}{\partial x_j}\right| = \frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)}$$

is the Jacobian.

1.371 If $y_1, y_2, \ldots\ldots\ldots, y_n$ are the partial derivatives of a function $F(x_1, x_2, \ldots\ldots, x_n)$:

$$y_i = \frac{\partial F}{\partial x_i} \quad (i = 1, 2, \ldots\ldots, n)$$

the symmetrical determinant:

$$H = \left|\frac{\partial^2 F}{\partial x_i\, \partial x_j}\right| = \frac{\partial\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \ldots\ldots, \frac{\partial F}{\partial x_n}\right)}{\partial(x_1, x_2, \ldots\ldots\ldots, x_n)}$$

is the Hessian.

1.372 If $y_1, y_2, \ldots\ldots, y_n$ are given as implicit functions of $x_1, x_2, \ldots\ldots\ldots, x_n$ by the n equations:

$$F_1(y_1, y_2, \ldots\ldots, y_n, x_1, x_2, \ldots\ldots\ldots, x_n) = 0$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$F_n(y_1, y_2, \ldots\ldots, y_n, x_1, x_2, \ldots\ldots\ldots, x_n) = 0$$

then

$$\frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)} \div \frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(y_1, y_2, \ldots, y_n)}$$

1.373 If the n functions $y_1, y_2, \ldots\ldots, y_n$ are not independent of each other the Jacobian, J, vanishes; and if $J = 0$ the n functions $y_1, y_2, \ldots\ldots, y_n$ are not independent of each other but are connected by a relation

$$F(y_1, y_2, \ldots\ldots\ldots, y_n) = 0$$

1.374 Covariant property. If the variables $x_1, x_2, \ldots, x_n$ are transformed by a linear substitution:

$$x_i = a_{i1}\xi_1 + a_{i2}\xi_2 + \ldots\ldots + a_{in}\xi_n \qquad (i = 1, 2, \ldots, n)$$

and the functions $y_1, y_2, \ldots\ldots, y_n$ of $x_1, x_2, \ldots\ldots, x_n$ become the functions $\eta_1, \eta_2, \ldots\ldots, \eta_n$ of $\xi_1, \xi_2, \ldots\ldots, \xi_n$:

$$J' = \frac{\partial(\eta_1, \eta_2, \ldots\ldots, \eta_n)}{\partial(\xi_1, \xi_2, \ldots\ldots, \xi_n)} = \frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)} \cdot |a_{ij}|$$

or

$$J' = J \cdot |a_{ij}|$$

where $|a_{ij}|$ is the determinant or modulus of the transformation.

For the Hessian,

$$H' = H \cdot |a_{ij}|^2.$$

1.380 To change the variables in a multiple integral:

$$I = \int \ldots\ldots \int F(y_1, y_2, \ldots\ldots, y_n)dy_1dy_2 \ldots\ldots dy_n$$

to new variables, $x_1, x_2, \ldots, x_n$ when $y_1, y_2, \ldots\ldots, y_n$ are given functions of $x_1, x_2, \ldots\ldots, x_n$:

$$I = \int \ldots\ldots \int \frac{\partial(y_1, y_2, \ldots\ldots, y_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)} F(x)dx_1dx_2 \ldots\ldots dx_n$$

where $F(x)$ is the result of substituting $x_1, x_2, \ldots, x_n$ for $y_1, y_2, \ldots, y_n$ in $F(y_1, y_2, \ldots\ldots, y_n)$.

PERMUTATIONS AND COMBINATIONS

1.400 Given n different elements. Represent each by a number, $1, 2, 3, \ldots\ldots, n$. The number of permutations of the n different elements is,

$$_nP_n = n!$$

e.g., $n = 3$:

$$(123),\ (132),\ (213),\ (231),\ (312),\ (321) = 6 = 3!$$

1.401 Given n different elements. The number of permutations in groups of r $(r<n)$, or the number of r-permutations, is,

$$_nP_r = \frac{n!}{(n-r)!}$$

e.g., $n = 4$, $r = 3$:

(123)(132)(124)(142)(134)(143)(234)(243)(231)(213)(214)(241)(341)(314)

1.402 Given n different elements. The number of ways they can be divided into m specified groups, with $x_1, x_2, \ldots\ldots, x_m$ in each group respectively, $(x_1 + x_2 + \ldots\ldots + x_m) = n$ is

$$\frac{n!}{x_1!x_2!\ldots\ldots x_m!}$$

e.g., $n = 6$, $m = 3$, $x_1 = 2$, $x_2 = 3$, $x_3 = 1$:

(12) (345) (6)	(13) (245) (6)	$\times 6 = 60$
(23) (145) (6)	(24) (135) (6)	
(34) (125) (6)	(35) (124) (6)	
(45) (123) (6)	(25) (234) (6)	
(14) (235) (6)	(15) (234) (6)	

1.403 Given n elements of which x_1 are of one kind, x_2 of a second kind, $\ldots\ldots$, x_m of an mth kind. The number of permutations is

$$\frac{n!}{x_1!x_2!\ldots\ldots x_m!}$$

$$x_1 + x_2 + \ldots\ldots + x_m = n.$$

1.404 Given n different elements. The number of ways they can be permuted among m specified groups, when blank groups are allowed, is

$$\frac{(m+n-1)!}{(m-1)!}$$

e.g., $n = 3$, $m = 2$:

(123,0)(132,0)(213,0)(231,0)(312,0)(321,0)(12,3)(21,3)(13,2)(31,2)(23,1)
(32,1)(1,23)(1,32)(2,31)(2,13)(3,12)(3,21)(0,123)(0,213)(0,132)(0,231)
(0,312)(0,321) = 24

1.405 Given n different elements. The number of ways they can be permuted among m specified groups, when blank groups are not allowed, so that each group contains at least one element, is

$$\frac{n!(n-1)!}{(n-m)!(m-1)!}$$

e.g., $n = 3$, $m = 2$:

(12,3)(21,3)(13,2)(31,2)(23,1)(32,1)(1,23)(1,32)(2,31)(2,13)(3,12)(3,21) = 12

1.406 Given n different elements. The number of ways they can be combined into m specified groups when blank groups are allowed is

$$m^n$$

e.g., $n = 3$, $m = 2$:

(123,0)(12,3)(13,2)(23,1)(1,23)(2,31)(3,12)(0,123) = 8

1.407 Given n similar elements. The number of ways they can be combined into m different groups when blank groups are allowed is

$$\frac{(n+m-1)!}{(m-1)!n!}$$

g., $n=6$, $m=3$:

roup 1	6	5	5	4	4	4	3	3	3	3	2	2	2	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	
roup 2	0	1	0	2	0	1	3	0	2	1	4	0	3	1	2	5	0	4	1	3	2	6	0	5	1	4	2	3	$= 28$
roup 3	0	0	1	0	2	1	0	3	1	2	0	4	1	3	2	0	5	1	4	2	3	0	6	1	5	2	4	3	

108 Given n similar elements. The number of ways they can be combined to m different groups when blank groups are not allowed, so that each group all contain at least one element, is

$$\frac{(n-1)!}{(m-1)!(n-m)!}.$$

BINOMIAL COEFFICIENTS

31

. $\displaystyle\binom{n}{k}=\frac{n!}{k!(n-k)!}=\binom{n}{n-k}={}_nC_k=\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}.$

. $\displaystyle\binom{n}{k}+\binom{n}{k+1}=\binom{n+1}{k+1}.$

. $\displaystyle\binom{n}{0}=1,\ \binom{n}{1}=n,\ \binom{n}{n}=1.$

. $\displaystyle\binom{-n}{k}=(-1)^k\binom{n+k-1}{k}.$

. $\displaystyle\binom{n}{k}=0$ if $n<k$.

. $\displaystyle\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\ldots\ldots+\binom{n}{k}=\binom{n+1}{k+1}.$

. $\displaystyle 1-\binom{n}{1}+\binom{n}{2}-\ldots\ldots+(-1)^k\binom{n}{k}=(-1)^k\binom{n-1}{k}.$

. $\displaystyle\binom{n}{k}+\binom{n}{k-1}\binom{r}{1}+\binom{n}{k-2}\binom{r}{2}+\ldots\ldots+\binom{r}{k}=\binom{n+r}{k}.$

. $\displaystyle 1+\binom{n}{1}+\binom{n}{2}+\ldots\ldots+\binom{n}{n}=2^n.$

. $\displaystyle 1-\binom{n}{1}+\binom{n}{2}-\ldots\ldots+(-1)^n\binom{n}{n}=0.$

. $\displaystyle 1+\binom{n}{1}^2+\binom{n}{2}^2+\ldots\ldots+\binom{n}{n}^2=\binom{2n}{n}.$

1.52 Table of Binomial Coefficients.

$$\binom{n}{1} = n.$$

$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$	$\binom{n}{11}$	$\binom{n}{12}$
1											
2	1										
3	3	1									
4	6	4	1								
5	10	10	5	1							
6	15	20	15	6	1						
7	21	35	35	21	7	1					
8	28	56	70	56	28	8	1				
9	36	84	126	126	84	36	9	1			
10	45	120	210	252	210	120	45	10	1		
11	55	165	330	462	462	330	165	55	11	1	
12	66	220	495	792	924	792	495	220	66	12	1

1.521 Glaisher, Mess. of Math. 47, p. 97, 1918, has given a complete table of binomial coefficients, from $n = 2$ to $n = 50$, and $k = 0$ to $k = n$.

1.61 Resolution into Partial Fractions.

If $F(x)$ and $f(x)$ are two polynomials in x and $f(x)$ is of higher degree than $F(x)$,

$$\frac{F(x)}{f(x)} = \sum \frac{F(a)}{\phi(a)} \frac{1}{x-a} + \sum \frac{1}{(p-1)!} \frac{d^{p-1}}{dc^{p-1}}\left[\frac{F(c)}{\phi(c)} \frac{1}{x-c}\right]$$

where

$$\phi(a) = \left[\frac{f(x)}{x-a}\right]_{x=a},$$

$$\phi(c) = \left[\frac{f(x)}{(x-c)^p}\right]_{x=c}.$$

The first summation is to be extended for all the simple roots, a, of $f(x)$ and the second summation for all the multiple roots, c, of order p, of $f(x)$.

FINITE DIFFERENCES AND SUMS.

1.811 Definitions.

1. $\Delta f(x) = f(x+h) - f(x)$.
2. $\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$.
$\qquad = f(x+2h) - 2f(x+h) + f(x)$.

3. $\Delta^3 f(x) = \Delta^2 f(x+h) - \Delta^2 f(x)$.
$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x).$$

. . . .

. . . .

4. $\Delta^n f(x) = f(x+nh) - \frac{n}{1} f(x+\overline{n-1}h) + \frac{n(n-1)}{2!} f(x+\overline{n-2}h) - \ldots$
$$+ (-1)^n f(x).$$

1.812

1. $\Delta[cf(x)] = c\Delta f(x)$ (c a constant).
2. $\Delta[f_1(x) + f_2(x) + \ldots] = \Delta f_1(x) + \Delta f_2(x) + \ldots$
3. $\Delta[f_1(x) \cdot f_2(x)] = f_1(x) \cdot \Delta f_2(x) + f_2(x+h) \cdot \Delta f_1(x)$
$$= f_1(x) \cdot \Delta f_2(x) + f_2(x) \cdot \Delta f_1(x) + \Delta f_1(x) \cdot \Delta f_2(x).$$
4. $\Delta \frac{f_1(x)}{f_2(x)} = \frac{f_2(x) \cdot \Delta f_1(x) - f_1(x) \cdot \Delta f_2(x)}{f_2(x) \cdot f_2(x+h)}$.

1.813 The nth difference of a polynomial of the nth degree is constant. If
$$f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$$
$$\Delta^n f(x) = n! a_0 h^n.$$

1.82

1. $\frac{\Delta^m [(x-b)(x-b-h)(x-b-2h) \ldots (x-b-\overline{n-1}h)]}{n(n-1)(n-2) \ldots (n-m+1)h^m}$
$$= (x-b)(x-b-h)(x-b-2h) \ldots (x-b-\overline{n-m-1}h).$$

2. $\Delta^m \frac{1}{(x+b)(x+b+h)(x+b+2h) \ldots (x+b+\overline{n-1}h)}$
$$= (-1)^m \frac{n(n+1)(n+2) \ldots (n+m-1)h^m}{(x+b)(x+b+h)(x+b+2h) \ldots (x+b+\overline{n+m-1}h)}.$$

3. $\Delta^m a^x = (a^h - 1)^m a^x$.

4. $\Delta \log f(x) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right)$.

5. $\Delta^m \sin(cx+d) = \left(2\sin\frac{ch}{2}\right)^m \sin\left(cx+d+m\frac{ch+\pi}{2}\right)$.

6. $\Delta^m \cos(cx+d) = \left(2\sin\frac{ch}{2}\right)^m \cos\left(cx+d+m\frac{ch+\pi}{2}\right)$.

1.83 Newton's Interpolation Formula.

$$f(x) = f(a) + \frac{x-a}{h}\Delta f(a) + \frac{(x-a)(x-a-h)}{2!\,h^2}\Delta^2 f(a) +$$

$$+ \frac{(x-a)(x-a-h)(x-a-2h)}{3!\,h^3}\Delta^3 f(a) + \ldots\ldots$$

$$+ \frac{(x-a)(x-a-h)\ldots\ldots(x-a-\overline{n-1}h)}{n!\,h^n}\Delta^n f(a)$$

$$+ \frac{(x-a)(x-a-h)\ldots\ldots(x-a-nh)}{\overline{n+1}!} f^{(n+1)}(\xi)$$

where ξ has a value intermediate between the greatest and least of a, $(a+nh)$, and x.

1.831

$$f(a+nh) = f(a) + \frac{n}{1!}\Delta f(a) + \frac{n(n-1)}{2!}\Delta^2 f(a) + \frac{n(n-1)(n-2)}{3!}\Delta^3 f(a)$$
$$+ \ldots\ldots + n\Delta^{n-1}f(a) + \Delta^n f(a).$$

1.832 Symbolically

1. $\Delta = e^{h\frac{\partial}{\partial x}} - 1$

2. $f(a+nh) = (1+\Delta)^n f(a)$

1.833 If $u_0 = f(a)$, $u_1 = f(a+h)$, $u_2 = f(a+2h)$, $\ldots$, $u_x = f(a+xh)$,

$$u_x = (1+\Delta)^x u_0 = e^{hx\frac{\partial}{\partial x}} u_0.$$

1.840 The operator inverse to the difference, Δ, is the sum, Σ.

$$\Sigma = \Delta^{-1} = \frac{1}{e^{\lambda\frac{\partial}{\partial x}} - 1},$$

1.841 If $\Delta F(x) = f(x)$,

$$\Sigma f(x) = F(x) + C,$$

where C is an arbitrary constant.

1.842

1. $\Sigma cf(x) = c\Sigma f(x)$.
2. $\Sigma[f_1(x) + f_2(x) + \ldots] = \Sigma f_1(x) + \Sigma f_2(x) + \ldots$
3. $\Sigma[f_1(x)\cdot\Delta f_2(x)] = f_1(x)\cdot f_2(x) - \Sigma[f_2(x+h)\cdot\Delta f_1(x)]$.

1.843 Indefinite Sums.

1. $$\sum[(x-b)(x-b-h)(x-b-2h)\ \ldots\ (x-b-\overline{n-1}h)]$$
$$=\frac{1}{(n+1)h}(x-b)(x-b-h)\ \ldots\ldots\ (x-b-nh)+C.$$

2. $$\sum\frac{1}{(x+b)(x+b+h)\ \ldots\ldots\ (x+b+\overline{n-1}h)}$$
$$=-\frac{1}{(n-1)h}\,\frac{1}{(x+b)(x+b+h)\ \ldots\ldots\ (x+b+\overline{n-2}h)}+C.$$

3. $$\sum a^x=\frac{a^x}{a^h-1}+C.$$

4. $$\sum\cos(cx+d)=\frac{\sin\left(cx-\frac{ch}{2}+d\right)}{2\sin\frac{ch}{2}}+C.$$

5. $$\sum\sin(cx+d)=-\frac{\cos\left(cx-\frac{ch}{2}+d\right)}{2\sin\frac{ch}{2}}+C.$$

1.844 If $f(x)$ is a polynomial of degree n,

$$\sum a^x f(x)=\frac{a^x}{a^h-1}\left\{f(x)-\frac{a^h}{a^h-1}\Delta f(x)+\left(\frac{a^h}{a^h-1}\right)^2\Delta^2 f(x)-\ldots\ldots\right.$$
$$\left.+\left(\frac{-a^h}{a^h-1}\right)^n\Delta^n f(x)\right\}+C.$$

1.845 If $f(x)$ is a polynomial of degree n,

$$f(x)=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n,$$

and

$$\sum f(x)=F(x)+C,$$
$$F(x)=c_0x^{n+1}+c_1x^n+c_2x^{n-1}+\ldots\ldots+c_nx+c_{n+1},$$

where

$$(n+1)hc_0=a_0$$
$$\frac{(n+1)n}{2!}h^2c_0+nhc_1=a_1$$
$$\frac{(n+1)n(n-1)}{3!}h^3c_0+\frac{n(n-1)}{2!}h^2c_1+(n-1)hc_2=a_2$$

. . . .

. . . .

The coefficient c_{n+1} may be taken arbitrarily.

1.850 Definite Sums. From the indefinite sum,

$$\Sigma f(x) = F(x) + C,$$

a definite sum is obtained by subtraction,

$$\sum_{a+mh}^{a+nh} f(x) = F(a+nh) - F(a+mh).$$

1.851

$$\sum_{a}^{a+nh} f(x) = f(a) + f(a+h) + f(a+2h) + \dots + f(a+\overline{n-1}h)$$
$$= F(a+nh) - F(a).$$

By means of this formula many finite sums may be evaluated.

1.852

$$\sum_{a}^{a+nh} (x-b)(x-b-h)(x-b-2h) \dots (x-b-\overline{k-1}h)$$
$$= \frac{(a-b+nh)(a-b+\overline{n-1}h) \dots (a-b+\overline{n-k}h)}{(k+1)h}$$
$$- \frac{(a-b)(a-b-h) \dots (a-b-kh)}{(k+1)h}.$$

1.853

$$\sum_{a}^{a+nh} (x-a)(x-a-h) \dots (x-a-\overline{k-1}h)$$
$$= \frac{n(n-1)(n-2) \dots (n-k)}{(k+1)} h^k.$$

1.854 If $f(x)$ is a polynomial of degree m it can be expressed:

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)(x-a-h) + \dots$$
$$+ A_m(x-a)(x-a-h) \dots (x-a-\overline{m-1}h),$$

$$\sum_{a}^{a+nh} f(x) = A_0 n + A_1 \frac{n(n-1)}{2} h + A_2 \frac{n(n-1)(n-2)}{3} h^2$$
$$+ A_m \frac{n(n-1) \dots (n-m)}{(m+1)} h^m.$$

855 If $f(x)$ is a polynomial of degree $(m-1)$ or lower, it can be expressed:

$$f(x) = A_0 + A_1(x+mh) + A_2(x+mh)(x+\overline{m-1}h)$$
$$+ \dots + A_{m-1}(x+mh) \dots (x+2h)$$

$$\frac{\qquad}{\dots (x+mh)} = \frac{A_0}{mh}\left\{ \frac{1}{a(a+h) \dots (a+\overline{m-1}h)} \right.$$

$$- \frac{1}{(a+nh)\ \dots\dots\ (a+\overline{n+m-1}h)}\Bigg\}$$

$$+ \frac{A_1}{(m-1)h}\left\{\frac{1}{a(a+h)\dots(a+\overline{m-2}h)} - \frac{1}{(a+nh)\dots(a+\overline{n+m-2}h)}\right\}$$

$$+\ \dots\dots\ + \frac{A_{m-1}}{h}\left\{\frac{1}{a} - \frac{1}{a+nh}\right\}.$$

1.856 If $f(x)$ is a polynomial of degree m it can be expressed:

$$f(x) = A_0 + A_1(x+mh) + A_2(x+mh)(x+\overline{m-1}h) + \dots\dots + A_m(x+mh)\ \dots\ (x+h)$$

and,

$$\sum_a^{a+nh}\frac{f(x)}{x(x+h)\ \dots\ (x+mh)} = \frac{A_0}{mh}\left\{\frac{1}{a(a+h)\ \dots\dots\ (a+\overline{m-1}h)}\right.$$

$$\left. - \frac{1}{(a+nh)\ \dots\dots\ (a+\overline{m+n-1}h)}\right\}$$

$$+\ \dots\dots\ + \frac{A_{m-1}}{h}\left\{\frac{1}{a} - \frac{1}{a+nh}\right\} + A_m\sum_a^{a+nh}\frac{1}{x}$$

where,

$$\sum_a^{a+nh}\frac{1}{x} = \frac{1}{a} + \frac{1}{a+h} + \frac{1}{a+2h} + \dots\dots + \frac{1}{a+\overline{n-1}h}.$$

1.86 Euler's Summation Formula.

$$\sum_a^b f(x) = \frac{1}{h}\int_a^b f(z)dz + A_1\left\{f(b) - f(a)\right\} + A_2 h\left\{f'(b) - f'(a)\right\}$$

$$+\ \dots\dots\ + A_{m-1}h^{m-2}\{f^{(m-2)}(b) - f^{(m-2)}(a)\}$$

$$- \int_0^h \phi_m(z)\sum_{x=a}^{x=b-h}\frac{d^m f(x+h-z)}{h\,dx^m}dz$$

$$\phi_m(z) = \frac{z^m}{m!} + A_1\frac{hz^{m-1}}{(m-1)!} + A_2\frac{h^2z^{m-2}}{(m-2)!} + \dots\dots + A_{m-1}h^{m-1}z.$$

$m!\phi_m(z)$, with $h = 1$, is the Bernoullian polynomial.

$A_1 = -\frac{1}{2}$, $A_{2k+1} = 0$; the coefficients A_{2k} are connected with Bernoulli's numbers (**6.902**), B_k, by the relation,

$$A_{2k} = (-1)^{k+1}\frac{B_k}{(2k)!}$$

1.861

$$\sum_a^b f(x) = \frac{1}{h}\int_a^b f(z)dz - \frac{1}{2}\left\{ f(b) - f(a) \right\} + \frac{h}{12}\left\{ f'(b) - f'(a) \right\}$$

$$- \frac{h^3}{720}\left\{ f'''(b) - f'''(a) \right\} + \frac{h^5}{30240}\left\{ f^{\text{v}}(b) - f^{\text{v}}(a) \right\} - \ldots\ldots$$

1.862

$$\sum u_x = C + \int u_x dx - \frac{1}{2}u_x + \frac{1}{12}\frac{du_x}{dx} - \frac{1}{720}\frac{d^3u_x}{dx^3} + \frac{1}{30240}\frac{d^5u_x}{dx^5} \ldots\ldots$$

SPECIAL FINITE SERIES

1.871 Arithmetical progressions. If s is the sum, a the first term, δ the common difference, l the last term, and n the number of terms,

$$s = a + (a + \delta) + (a + 2\delta) + \ldots\ldots [a + (n - 1)\delta]$$

$$l = a + (n - 1)\,\delta$$

$$s = \frac{n}{2}[2a + (n - 1)\delta]$$

$$= \frac{n}{2}(a + l).$$

1.872 Geometrical progressions.

$$s = a + ap + ap^2 + \ldots\ldots + ap^{n-1}$$

$$s = a\frac{p^n - 1}{p - 1}$$

If $p < 1$, $n = \infty$, $s = \dfrac{a}{1 - p}$.

1.873 Harmonical progressions. $a, b, c, d, \ldots\ldots$ form an harmonical progression if the reciprocals, $1/a$, $1/b$, $1/c$, $1/d$, $\ldots\ldots$ form an arithmetical progression.

1.874.

1. $$\sum_{x=1}^{x=n} x = \frac{n(n+1)}{2}$$

2. $$\sum_{x=1}^{x=n} x^2 = \frac{n(n+1)\,(2n+1)}{6}$$

3. $$\sum_{x=1}^{x=n} x^3 = \left[\frac{n(n+1)}{2}\right]^2$$

4. $$\sum_{x=1}^{x=n} x^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

1.875 In general,

$$\sum_{x=1}^{x=n} x^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + \frac{1}{2}\binom{k}{1} B_1 n^{k-1} - \frac{1}{4}\binom{k}{3} B_2 n^{k-3} + \frac{1}{6}\binom{k}{5} B_3 n^{k-5} - \dots$$

B_1, B_2, B_3, ... are Bernoulli's numbers (6.902), $\binom{k}{h}$ are the binomial coefficients (1.51); the series ends with the term in n if k is even, and with the term in n^2 if k is odd.

1.876

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} = \gamma + \log n + \frac{1}{2n} - \frac{a_2}{n(n+1)}$$

$$- \frac{a_3}{n(n+1)(n+2)} - \dots,$$

γ = Euler's constant = 0.5772156649 ...

$$a_2 = \frac{1}{12}$$

$$a_3 = \frac{1}{12}$$

$$a_4 = \frac{19}{80} \qquad a_k = \frac{1}{k}\int_0^1 x(1-x)(2-x)\dots\dots(k-1-x)dx$$

$$a_5 = \frac{9}{20}$$

1.877

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} = \frac{\pi^2}{6} - \frac{b_1}{n+1} - \frac{b_2}{(n+1)(n+2)}$$

$$- \frac{b_3}{(n+1)(n+2)(n+3)} - \dots,$$

$$b_k = \frac{(k-1)!}{k}$$

1.878

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots + \frac{1}{n^3} = C - \frac{c_2}{(n+1)(n+2)}$$

$$- \frac{c_3}{(n+1)(n+2)(n+3)} - \dots$$

$$C = \sum^{\infty} \frac{1}{k^3} = 1.2020569032$$

1.879 Stirling's Formula.

$$\log(n!) = \log\sqrt{2\pi} + \left(n + \frac{1}{2}\right)\log n - n$$
$$+ \frac{A_2}{n} + \ldots\ldots + A_{2k-2}\frac{(2k-4)!}{n^{2k-3}}$$
$$+ \theta A_{2k}\frac{(2k-2)!}{n^{2k-1}}$$

$0<\theta<1$. The coefficients A_k are given in 1.86.

1.88

1. $1 + 1! + 2\cdot 2! + 3\cdot 3! + \ldots\ldots + n\cdot n! = (n+1)!$

2. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots\ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$

3. $1\cdot 2\cdot 3 \ldots\ldots r + 2\cdot 3\cdot 4 \ldots (r+1) + \ldots\ldots\ldots\ldots + n(n+1)(n+2)$
$\ldots\ldots (n+r-1)$

$$= \frac{n(n+1)(n+2)\ldots\ldots(n+r)}{r+1}.$$

4. $1\cdot p + 2(p+1) + 3(p+2) + \ldots\ldots\ldots\ldots + n(p+n-1)$

$$= \frac{1}{6}n(n+1)(3p+2n-2).$$

5. $p\cdot q + (p-1)(q-1) + (p-2)(q-2) + \ldots\ldots\ldots\ldots (p-n)(q-n)$

$$= \frac{1}{6}n[6pq - (n-1)(3p+3q-2n+1)].$$

6. $1 + \frac{b}{a} + \frac{b(b+1)}{a(a+1)} + \ldots\ldots + \frac{b(b+1)\ldots\ldots(b+n-1)}{a(a+1)\ldots\ldots(a+n-1)}.$

$$= \frac{b(b+1)\ldots\ldots(b+n)}{(b+1-a)a(a+1)\ldots\ldots(a+n-1)} - \frac{a-1}{b+1-a}.$$

II. GEOMETRY

2.00 Transformation of coördinates in a plane.

2.001 Change of origin. Let x, y be a system of *rectangular* or *oblique* coördinates with origin at O. Referred to x, y the coördinates of the new origin O' are a, b. Then referred to a parallel system of coördinates with origin at O' the coördinates are x', y'.

$$x = x' + a$$
$$y = y' + b.$$

2.002 Origin unchanged. Directions of axes changed. Oblique coördinates. Let ω be the angle between the x- y axes measured counter-clockwise from the x- to the y-axis. Let the x'-axis make an angle α with the x-axis and the y'-axis an angle β with the x-axis. All angles are measured counter-clockwise from the x-axis. Then

$$x \sin \omega = x' \sin (\omega - \alpha) + y' \sin (\omega - \beta)$$
$$y \sin \omega = x' \sin \alpha + y' \sin \beta$$
$$\omega' = \beta - \alpha.$$

2.003 Rectangular axes. Let both new and old axes be rectangular, the new axes being turned through an angle θ with respect to the old axes. Then $\omega = \frac{\pi}{2}$, $\alpha = \theta$, $\beta = \frac{\pi}{2} + \theta$.

$$x = x' \cos \theta - y' \sin \theta$$
$$y = x' \sin \theta + y' \cos \theta.$$

2.010 Polar coördinates. Let the y-axis make an angle ω with the x-axis and let the x-axis be the initial line for a system of polar coördinates r, θ. All angles are measured in a counter-clockwise direction from the x-axis.

$$x = \frac{r \sin (\omega - \theta)}{\sin \omega}$$

$$y = r \frac{\sin \theta}{\sin \omega}.$$

2.011 If the x, y axes are rectangular, $\omega = \frac{\pi}{2}$,

$$x = r \cos \theta$$
$$y = r \sin \theta$$

2.020 Transformation of coördinates in three dimensions.

2.021 Change of origin. Let x, y, z be a system of *rectangular* or *oblique* coördinates with origin at O. Referred to x, y, z the coördinates of the new origin O' are a, b, c. Then referred to a parallel system of coördinates with origin at O' the coördinates are x', y', z'.

$$x = x' + a$$
$$y = y' + b$$
$$z = z' + c$$

2.022 Transformation from one to another rectangular system. Origin unchanged. The two systems are x, y, z and x' y' z'.

Referred to x, y, z the direction cosines of x' are l_1, m_1, n_1
Referred to x, y, z the direction cosines of y' are l_2, m_2, n_2
Referred to x, y, z the direction cosines of z' are l_3, m_3, n_3

The two systems are connected by the scheme:

	x'	y'	z'
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

$$x = l_1x' + l_2y' + l_3z' \qquad x' = l_1x + m_1y + n_1z$$
$$y = m_1x' + m_2y' + m_3z' \qquad y' = l_2x + m_2y + n_2z$$
$$z = n_1x' + n_2y' + n_3z' \qquad z' = l_3x + m_3y + n_3z$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \qquad l_1^2 + l_2^2 + l_3^2 = 1$$
$$l_2^2 + m_2^2 + n_2^2 = 1 \qquad m_1^2 + m_2^2 + m_3^2 = 1$$
$$l_3^2 + m_3^2 + n_3^2 = 1 \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0 \qquad l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0 \qquad l_2l_3 + m_2m_3 + n_2n_3 = 0$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \qquad l_3l_1 + m_3m_1 + n_3n_1 = 0$$

2.023 If the transformation from one to another rectangular system is a rotation through an angle θ about an axis which makes angles α, β, γ with x, y, z respectively,

$$2\cos\theta = l_1 + m_2 + n_3 - 1$$

$$\frac{\cos^2\alpha}{m_2+n_3-l_1-1}=\frac{\cos^2\beta}{n_3+l_1-m_2-1}=\frac{\cos^2\gamma}{l_1+m_2-n_3-1}$$

2.024 Transformation from a rectangular to an oblique system. x, y, z rectangular system: x', y', z' oblique system.

$$\begin{array}{lll}
\cos\widehat{xx'}=l_1 & \cos\widehat{xy'}=l_2 & \cos\widehat{xz'}=l_3\\
\cos\widehat{yx'}=m_1 & \cos\widehat{yy'}=m_2 & \cos\widehat{yz'}=m_3\\
\cos\widehat{zx'}=n_1 & \cos\widehat{zy'}=n_2 & \cos\widehat{zz'}=n_3
\end{array}$$

$$\begin{aligned}
x&=l_1x'+l_2y'+l_3z'\\
y&=m_1x'+m_2y'+m_3z'\\
z&=n_1x'+n_2y'+n_3z'
\end{aligned}$$

$$\begin{aligned}
\cos\widehat{y'z'}&=l_2l_3+m_2m_3+n_2n_3\\
\cos\widehat{z'x'}&=l_3l_1+m_3m_1+n_3n_1\\
\cos\widehat{x'y'}&=l_1l_2+m_1m_2+n_1n_2
\end{aligned}$$

$$\begin{aligned}
l_1^2+m_1^2+n_1^2&=1\\
l_2^2+m_2^2+n_2^2&=1\\
l_3^2+m_3^2+n_3^2&=1
\end{aligned}$$

2.025 Transformation from one to another oblique system.

$$\begin{array}{lll}
\cos\widehat{xx'}=l_1 & \cos\widehat{xy'}=l_2 & \cos\widehat{xz'}=l_3\\
\cos\widehat{yx'}=m_1 & \cos\widehat{yy'}=m_2 & \cos\widehat{yz'}=m_3\\
\cos\widehat{zx'}=n_1 & \cos\widehat{zy'}=n_2 & \cos\widehat{zz'}=n_3
\end{array}$$

$$\Delta=\begin{vmatrix} l_1 & l_2 & l_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{vmatrix}$$

$$\begin{aligned}
x&=l_1x'+l_2y'+l_3z'\\
y&=m_1x'+m_2y'+m_3z'\\
z&=n_1x'+n_2y'+n_3z'
\end{aligned}$$

$$\begin{aligned}
\Delta\cdot x'&=(m_2n_3-m_3n_2)x+(n_2l_3-n_3l_2)y+(l_2m_3-l_3m_2)z,\\
\Delta\cdot y'&=(m_3n_1-m_1n_3)x+(n_3l_1-n_1l_3)y+(l_3m_1-l_1m_3)z,\\
\Delta\cdot z'&=(m_1n_2-m_2n_1)x+(n_1l_2-n_2l_1)y+(l_1m_2-l_2m_1)z.
\end{aligned}$$

$$\begin{aligned}
l_1^2+m_1^2+n_1^2+2m_1n_1\cos\widehat{yz}+2n_1l_1\cos\widehat{zx}+2l_1m_1\cos\widehat{xy}&=1,\\
l_2^2+m_2^2+n_2^2+2m_2n_2\cos\widehat{yz}+2n_2l_2\cos\widehat{zx}+2l_2m_2\cos\widehat{xy}&=1,\\
l_3^2+m_3^2+n_3^2+2m_3n_3\cos\widehat{yz}+2n_3l_3\cos\widehat{zx}+2l_3m_3\cos\widehat{xy}&=1.
\end{aligned}$$

$$\begin{aligned}
x+y\cos\widehat{xy}+z\cos\widehat{xz}&=l_1x'+l_2y'+l_3z',\\
y+x\cos\widehat{xy}+z\cos\widehat{zy}&=m_1x'+m_2y'+m_3z',
\end{aligned}$$

2.026 Transformation from one to another oblique system.

If n_x, n_y, n_z are the normals to the planes yz, zx, xy and n_x', n_y', n_z' the normals to the planes $y'z'$, $z'x'$, $x'y'$,

$$x \cos \widehat{xn_x} = x' \cos \widehat{x'n_x} + y' \cos \widehat{y'n_x} + z' \cos \widehat{z'n_x}.$$
$$y \cos \widehat{yn_y} = x' \cos \widehat{x'n_y} + y' \cos \widehat{y'n_y} + z' \cos \widehat{z'n_y}.$$
$$z \cos \widehat{zn_z} = x' \cos \widehat{x'n_z} + y' \cos \widehat{y'n_z} + z' \cos \widehat{z'n_z}.$$

$$x' \cos \widehat{x'n_x'} = x \cos \widehat{xn_x'} + y \cos \widehat{yn_x'} + z \cos \widehat{zn_x'}.$$
$$y' \cos \widehat{y'n_y'} = x \cos \widehat{xn_y'} + y \cos \widehat{yn_y'} + z \cos \widehat{zn_y'}.$$
$$z' \cos \widehat{z'n_z'} = x \cos \widehat{xn_z'} + y \cos \widehat{yn_z'} + z \cos \widehat{zn_z'}.$$

2.030 Transformation from rectangular to spherical polar coördinates.

r, the radius vector to a point makes an angle θ with the z-axis, the projection of r on the x-y plane makes an angle ϕ with the x-axis.

$$x = r \sin \theta \cos \phi \qquad r^2 = x^2 + y^2 + z^2$$
$$y = r \sin \theta \sin \phi \qquad \theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$
$$z = r \cos \theta \qquad \phi = \tan^{-1} \frac{y}{x}$$

2.031 Transformation from rectangular to cylindrical coördinates.

ρ, the perpendicular from the z-axis to a point makes an angle θ with the x-z plane.

$$x = \rho \cos \theta \qquad \rho = \sqrt{x^2 + y^2}$$
$$y = \rho \sin \theta \qquad \theta = \tan^{-1} \frac{y}{x}$$
$$z = z$$

2.032 Curvilinear coördinates in general.

See **4.0**

2.040 Eulerian Angles.

$Oxyz$ and $Ox'y'z'$ are two systems of rectangular axes with the same origin O. OK is perpendicular to the plane zOz' drawn so that if Oz is vertical, and the projection of Oz' perpendicular to Oz is directed to the south, then OK is directed to the east.

$$\text{Angles} \quad \widehat{z'Oz} = \theta,$$
$$\widehat{yOK} = \phi,$$
$$\widehat{y'OK} = \psi,$$

The direction cosines of the two systems of axes are given by the following scheme:

	x	y	z
x'	$\cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi$	$\sin\phi\cos\theta\cos\psi + \cos\phi\sin\psi$	$-\sin\theta\cos\psi$
y'	$-\cos\phi\cos\theta\sin\psi - \sin\phi\cos\psi$	$-\sin\phi\cos\theta\sin\psi + \cos\phi\cos\psi$	$\sin\theta\sin\psi$
z'	$\cos\phi\sin\theta$	$\sin\phi\sin\theta$	$\cos\theta$

2.050 Trilinear Coördinates.

A point in a plane is defined if its distances from two intersecting lines are given. Let CA, CB (Fig. 1) be these lines:

$$PR = p, \quad PS = q, \quad PT = r.$$

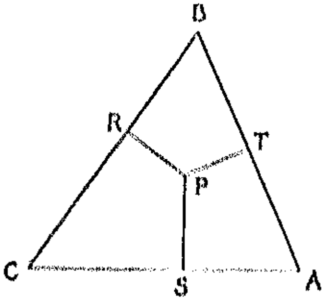

FIG. 1

Taking CA and CB as the x-, y-axes, including an angle C,

$$x = \frac{p}{\sin C},$$

$$y = \frac{q}{\sin C}.$$

Any curve $f(x,y) = 0$ becomes:

$$f\left(\frac{p}{\sin C}, \frac{q}{\sin C}\right) = 0.$$

If Δ is the area of the triangle CAB (triangle of reference),

$$2\Delta = ap + bq + cr,$$
$$a = BC,$$
$$b = CA,$$
$$c = AB,$$

and the equation of a curve may be written in the homogeneous form:

$$f\left(\frac{2\Delta p}{(ap + bq + cr)\sin C}, \frac{2\Delta q}{(ap + bq + cr)\sin C}\right) = 0.$$

2.060 Quadriplanar Coördinates.

These are the analogue in 3 dimensions of trilinear coördinates in a plane (2.050).

x_1, x_2, x_3, x_4 denote the distances of a point P from the four sides of a tetrahedron (the tetrahedron of reference); l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3; and l_4, m_4, n_4 the direction cosines of the normals to the planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ with respect to a rectangular system of coördinates x, y, z; and d_1, d_2, d_3, d_4 the distances of these 4 planes from the origin of coördinates:

$$(1)\begin{cases} x_1 = l_1x + m_1y + n_1z - d_1 \\ x_2 = l_2x + m_2y + n_2z - d_2 \\ x_3 = l_3x + m_3y + n_3z - d_3 \\ x_4 = l_4x + m_4y + n_4z - d_4. \end{cases}$$

s_1, s_2, s_3, and s_4 are the areas of the 4 faces of the tetrahedron of reference and V its volume:

$$3V = x_1s_1 + x_2s_2 + x_3s_3 + x_4s_4.$$

By means of the first 3 equations of (1) x, y, z are determined:

$$\begin{aligned} x &= A_1x_1 + B_1x_2 + C_1x_3 + D_1, \\ y &= A_2x_1 + B_2x_2 + C_2x_3 + D_2, \\ z &= A_3x_1 + B_3x_2 + C_3x_3 + D_3. \end{aligned}$$

The equation of any-surface,

$$F(x,y,z) = 0,$$

may be written in the homogeneous form:

$$\begin{aligned} F\Big\{&\Big[A_1x_1 + B_1x_2 + C_1x_3 + \frac{D_1}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\Big], \\ &\Big[A_2x_1 + B_2x_2 + C_2x_3 + \frac{D_2}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\Big], \\ &\Big[A_3x_1 + B_3x_2 + C_3x_3 + \frac{D_3}{3V}(s_1x_1 + s_2x_2 + s_3x_3 + s_4x_4)\Big]\Big\} = 0. \end{aligned}$$

PLANE GEOMETRY

2.100 The equation of a line:

$$Ax + By + C = 0.$$

2.101 If p is the perpendicular from the origin upon the line, and α and β the angles p makes with the x- and y-axes:

$$p = x\cos\alpha + y\cos\beta.$$

2.102 If α' and β' are the angles the line makes with the x- and y-axes:

$$p = y\cos\alpha' - x\cos\beta'.$$

2.103 The equation of a line may be written

$$y = ax + b.$$

a = tangent of angle the line makes with the x-axis,

2.104 The two lines:

$$y = a_1x + b_1,$$
$$y = a_2x + b_2,$$

intersect at the point:

$$x = \frac{b_2 - b_1}{a_1 - a_2}, \qquad y = \frac{a_1b_2 - a_2b_1}{a_1 - a_2}.$$

2.105 If ϕ is the angle between the two lines **2.104**:

$$\tan\phi = \pm\frac{a_1 - a_2}{1 + a_1a_2}.$$

2.106 Equations of two parallel lines:

$$\begin{cases} Ax + By + C_1 = 0 \\ Ax + By + C_2 = 0 \end{cases} \quad\text{or}\quad \begin{cases} y = ax + b_1 \\ y = ax + b_2. \end{cases}$$

2.107 Equations of two perpendicular lines:

$$\begin{cases} Ax + By + C_1 = 0 \\ Bx - Ay + C_2 = 0 \end{cases} \quad\text{or}\quad \begin{cases} y = ax + b_1, \\ y = -\dfrac{x}{a} + b_2. \end{cases}$$

2.108 Equation of line through x_1, y_1 and parallel to the line:

$$Ax + By + C = 0 \qquad\text{or}\qquad y = ax + b,$$
$$A(x - x_1) + B(y - y_1) = 0 \qquad\text{or}\qquad y - y_1 = a(x - x_1).$$

2.109 Equation of line through x_1, y_1 and perpendicular to the line

$$Ax + By + C = 0 \qquad\text{or}\qquad y = ax + b,$$
$$B(x - x_1) - A(y - y_1) = 0 \qquad\text{or}\qquad y - y_1 = -\frac{x - x_1}{a}.$$

2.110 Equation of line through x_1, y_1 making an angle ϕ with the line $y = ax + b$:

$$y - y_1 = \frac{a + \tan\phi}{1 - a\tan\phi}(x - x_1).$$

2.111 Equation of line through the two points, x_1, y_1 and x_2, y_2:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

2.112 Perpendicular distance from the point x_1, y_1 to the line

$$Ax + By + C = 0 \qquad\text{or}\qquad y = ax + b,$$
$$p = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \qquad\text{or}\qquad p = \frac{y_1 - ax_1 - b}{\sqrt{1 + a^2}}.$$

2.113 Polar equation of the line $y = ax + b$:

$$r = \frac{b\cos\alpha}{\sin(\theta - \alpha)},$$

where

2.114 If p, the perpendicular to the line from the origin, makes an angle β with the axis:

$$p = r\cos(\theta - \beta).$$

2.130 Area of polygon whose vertices are at $x_1, y_1;\ x_2, y_2;\ \ldots\ldots\ldots,\ x_n, y_n = A.$

$$2A = y_1(x_n - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + \ldots\ldots\ldots + y_n(x_{n-1} - x_1).$$

PLANE CURVES

2.200 The equation of a plane curve in rectangular coördinates may be given in the forms:

(a) $y = f(x)$.

(b) $x = f_1(t),\ y = f_2(t)$. The parametric form.

(c) $F(x,y) = 0$.

2.201 If τ is the angle between the tangent to the curve and the x-axis:

(a) $\tan\tau = \dfrac{dy}{dx} = y'$.

(b) $\tan\tau = \dfrac{\dfrac{df_2(t)}{dt}}{\dfrac{df_1(t)}{dt}}$.

(c) $\tan\tau = -\dfrac{\dfrac{\partial F(x,y)}{\partial x}}{\dfrac{\partial F(x,y)}{\partial y}}$.

In the following formulas,

$y' = \dfrac{dy}{dx} = \tan\tau$ (2.201).

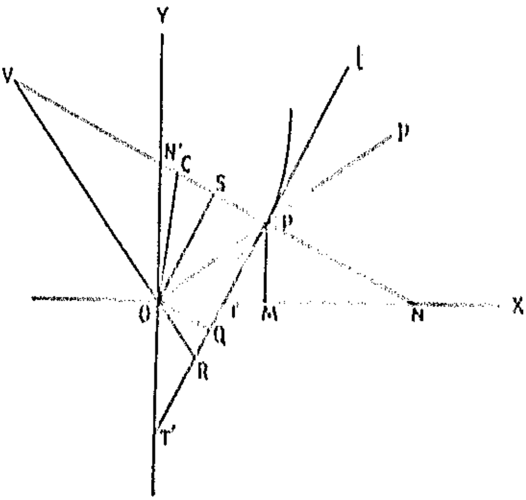

FIG. 2

2.202 $OM = x,\ MP = y$, angle $XTP = \tau$.

$$TP = y\csc\tau = \frac{y\sqrt{1+y'^2}}{y'} = \text{tangent},$$

$$TM = y\cot\tau = \frac{y}{y'} = \text{subtangent},$$

$$PN = y\sec\tau = y\sqrt{1+y'^2} = \text{normal},$$

$$MN = y\tan\tau = yy' = \text{subnormal}.$$

2.203 $OT = x - \dfrac{y}{y'} =$ intercept of tangent on x-axis,

$OT' = y - xy' =$ intercept of tangent on y-axis,

$ON = x + yy' =$ intercept of normal on x-axis,

$ON' = y + \dfrac{x}{y'} =$ intercept of normal on y-axis.

2.204 $OQ = \dfrac{y - xy'}{\sqrt{1 + y'^2}} =$ distance of tangent from origin $= PS =$ projection of radius vector on normal.

$$\text{Coördinates of } Q: \quad \frac{y'(xy' - y)}{1 + y'^2}, \ \frac{y - xy'}{1 + y'^2}.$$

2.205 $OS = \dfrac{x + yy'}{\sqrt{1 + y'^2}} =$ distance of normal from origin $= PQ =$ projection of radius vector on tangent.

$$\text{Coördinates of } S: \quad \frac{x + yy'}{1 + y'^2}, \ \frac{(x + yy')y'}{1 + y'^2}.$$

2.206 $OR = \dfrac{\sqrt{x^2 + y^2}\,(y - xy')}{x + yy'} =$ polar subtangent,

$PR = \dfrac{(x^2 + y^2)\sqrt{1 + y'^2}}{x + yy'} =$ polar tangent,

$$\text{Coördinates of } R: \quad \frac{y(xy' - y)}{x + yy'}, \ \frac{x(y - xy')}{x + yy'}.$$

2.207 $OV = \dfrac{\sqrt{x^2 + y^2}\,(x + yy')}{y - xy'} =$ polar subnormal,

$PV = \dfrac{(x^2 + y^2)\sqrt{1 + y'^2}}{y - xy'} =$ polar normal,

$$\text{Coördinates of } V: \quad \frac{y(x + yy')}{y - xy'}, \ -\frac{x(x + yy')}{y - xy'}.$$

2.210 The equations of the tangent at x_1, y_1 to the curve in the three forms of **2.200** are:

(a) $$y - y_1 = f'(x_1)\,(x - x_1).$$

(b) $$(y - y_1)f_1'(t_1) = (x - x_1)f_2'(t_1).$$

(c) $$(x - x_1)\left(\frac{\partial F}{\partial x}\right)_{\substack{x=x_1\\y=y_1}} + (y - y_1)\left(\frac{\partial F}{\partial y}\right)_{\substack{x=x_1\\y=y_1}} = 0.$$

2.211 The equations of the normal at x_1, y_1 to the curve in the three forms of **2.200** are:

(a) $$f'(x_1)\,(y - y_1) + (x - x_1) = 0.$$

(b) $$(y - y_1)f_2'(t_1) + (x - x_1)f_1'(t_1) = 0.$$

(c) $$(x - x_1)\left(\frac{\partial F}{\partial y}\right)_{\substack{x=x_1\\y=y_1}} = (y - y_1)\left(\frac{\partial F}{\partial x}\right)_{\substack{x=x_1\\y=y_1}}.$$

2.212 The perpendicular from the origin upon the tangent to the curve $F(x, y) = 0$ at the point x, y is:

$$p = \frac{x\dfrac{\partial F}{\partial x} + y\dfrac{\partial F}{\partial y}}{\sqrt{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2}}.$$

2.213 Concavity and Convexity. If in the neighborhood of a point P a curve lies entirely on one side of the tangent, it is concave or convex upwards according as $y'' = \dfrac{d^2y}{dx^2}$ is positive or negative. The positive direction of the axes are shown in figure 2.

2.220 Convention as to signs. The positive direction of the normal is related to the positive direction of the tangent as the positive y-axis is related to the positive x-axis. The angle τ is measured positively in the counter-clockwise direction from the positive x-axis to the positive tangent.

2.221 Radius of curvature $= \rho$; curvature $= 1/\rho$.

$$\frac{1}{\rho} = \frac{d\tau}{ds},$$

where s is the arc drawn from a fixed point of the curve in the direction of the positive tangent.

2.222 Formulas for the radius of curvature of curves given in the three forms of **2.200**.

(a)
$$\rho = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}$$

(b)
$$\rho = \frac{\left\{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}} = \frac{\left(\dfrac{ds}{dt}\right)^3}{\left\{\left(\dfrac{d^2x}{dt^2}\right)^2 + \left(\dfrac{d^2y}{dt^2}\right)^2 - \left(\dfrac{d^2s}{dt^2}\right)^2\right\}^{\frac{1}{2}}}$$

If s is taken as the parameter t:

(b′)
$$\frac{1}{\rho} = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}^{\frac{1}{2}}$$

(c)
$$\rho = -\frac{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2 + \left(\dfrac{\partial F}{\partial y}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{\partial^2 F}{\partial x^2}\left(\dfrac{\partial F}{\partial y}\right)^2 - 2\dfrac{\partial^2 F}{\partial x\partial y}\dfrac{\partial F}{\partial x}\dfrac{\partial F}{\partial y} + \dfrac{\partial^2 F}{\partial y^2}\left(\dfrac{\partial F}{\partial x}\right)^2}$$

2.223 The center of curvature is a point C (fig. 2) on the normal at P such that $PC = \rho$. If ρ is positive C lies on the positive normal (**2.213**); if negative, on the negative normal.

2.224 The circle of curvature is a circle with C as center and radius $= \rho$.

2.225 The chord of curvature is the chord of the circle of curvature passing through the origin and the point P.

2.226 The coördinates of the center of curvature at the point x, y are ξ, η:

$$\xi = x - \rho \sin \tau \qquad\qquad \tan \tau = \frac{dy}{dx}$$
$$\eta = y + \rho \cos \tau$$

If l', m' are the direction cosines of the positive normal,

$$\xi = x + l'\rho$$
$$\eta = y + m'\rho.$$

2.227 If l, m are the direction cosines of the positive tangent and l', m' those of the positive normal,

$$\frac{dl}{ds} = \frac{l'}{\rho}, \quad \frac{dm}{ds} = \frac{m'}{\rho},$$
$$l' = -m, \quad m' = l,$$
$$\frac{dl'}{ds} = -\frac{l}{\rho}, \quad \frac{dm'}{ds} = -\frac{m}{\rho}.$$

2.228 If the tangent and normal at P are taken as the x- and y- axes, then

$$\rho = \underset{x=0}{\text{limit}} \frac{x^2}{2y}.$$

2.229 Points of Inflexion. For a curve given in the form (a) of **2.200** a point of inflexion is a point at which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ exists and is continuous and at which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ vanishes and changes sign.

If the curve is given in the form (b) a point of inflexion, t_1, is a point at which the determinant:

$$\begin{vmatrix} f_1''(t_1) & f_2''(t_1) \\ f_1'(t_1) & f_2'(t_1) \end{vmatrix}$$

vanishes and changes sign.

2.230 Eliminating x and y between the coördinates of the center of curvature (**2.226**) and the corresponding equations of the curve (**2.200**) gives the equation of the evolute of the curve = the locus of the center of curvature. A curve which has a given curve for evolute is called an involute of the given curve.

2.231 The envelope to a family of curves,

1. $$F(x, y, \alpha) = 0,$$

where α is a parameter, is obtained by eliminating α between (1) and

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

2.232 If the curve is given in the form,

1. $$x = f_1(t, \alpha)$$

2. $$y = f_2(t, \alpha),$$

the envelope is obtained by eliminating t and α between (1), (2) and the functional determinant,

3. $$\frac{\partial(f_1, f_2)}{\partial(t, \alpha)} = 0 \quad \text{(see } \mathbf{1.370}\text{)}$$

2.233 Pedal Curves. The locus of the foot of the perpendicular from a fixed point upon the tangent to a given curve is the pedal of the given curve with reference to the fixed point.

2.240 Asymptotes. The line

$$y = ax + b$$

is an asymptote to the curve $y = f(x)$ if

$$a = \underset{x \to \infty}{\text{limit}}\ f'(x)$$

$$b = \underset{x \to \infty}{\text{limit}}\ [f(x) - xf'(x)]$$

2.241 If the curve is

$$x = f_1(t),\ y = f_2(t),$$

and if for a value of t, t_1, f_1 or f_2 becomes infinite, there will be an asymptote if for that value of t the direction of the tangent to the curve approaches a limit and the distance of the tangent from a fixed point approaches a limit.

2.242 An asymptote may sometimes be determined by expanding the equation of the curve in a series,

$$y = \sum_{k=0}^{n} a_k x^k + \sum_{k=1}^{\infty} \frac{b_k}{x^k}.$$

If

$$\underset{x \to \infty}{\text{limit}} \sum_{k=1}^{\infty} \frac{b_k}{x^k} = 0,$$

the equation of the asymptote is

$$y = \sum_{k=0}^{n} a_k x^k$$

If of the first degree in x, this represents a rectilinear asymptote; if of a higher degree, a curvilinear asymptote.

2.250 Singular Points. If the equation of the curve is $F(x, y) = 0$, singular points are those for which

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0.$$

Put,

$$\Delta = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x\,\partial y}\right)^2.$$

If $\Delta < 0$ the singular point is a double point with two distinct tangents.

$\Delta > 0$ the singular point is an isolated point with no real branch of the curve through it.

$\Delta = 0$ the singular point is an osculating point, or a cusp. The curve has two branches, with a common tangent, which meet at the singular point.

If $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial^2 F}{\partial x^2}$, $\frac{\partial^2 F}{\partial y^2}$, $\frac{\partial^2 F}{\partial x\,\partial y}$ simultaneously vanish at a point the singular point is one of higher order.

PLANE CURVES, POLAR COÖRDINATES

2.270 The equation of the curve is given in the form,

$$r = f(\theta).$$

In figure 2, $OP = r$, angle $XOP = \theta$, angle $XTP = \tau$, angle $pPt = \phi$.

2.271 θ is measured in the counter-clockwise direction from the initial line, OX, and s, the arc, is so chosen as to increase with θ. The angle ϕ is measured in the counter-clockwise direction from the positive radius vector to the positive tangent. Then,

$$\tau = \theta + \phi.$$

2.272

$$\tan\phi = \frac{r\,d\theta}{dr}$$

$$\sin\phi = \frac{r\,d\theta}{ds}$$

$$\cos\phi = \frac{dr}{ds}$$

2.273
$$\tan \tau = \frac{\sin\theta \dfrac{dr}{d\theta} + r\cos\theta}{\cos\theta \dfrac{dr}{d\theta} - r\sin\theta}$$

$$ds = \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta$$

2.274
$$PR = r\sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2} = \text{polar tangent}$$

$$PV = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \text{polar normal}$$

$$OR = r^2\frac{d\theta}{dr} \qquad = \text{polar subtangent}$$

$$OV = \frac{dr}{d\theta} \qquad = \text{polar subnormal.}$$

2.275 $OQ = \dfrac{r^2}{\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}} = p =$ distance of tangent from origin.

$OS = \dfrac{r\dfrac{dr}{d\theta}}{\sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}} =$ distance of normal from origin.

2.276 If $u = \dfrac{1}{r}$, the curve $r = f(\theta)$ is concave or convex to the origin according as

$$u + \frac{d^2u}{d\theta^2}$$

is positive or negative. At a point of inflexion this quantity vanishes and changes sign.

2.280 The radius of curvature is,

$$\rho = \frac{\left\{ r^2 + \left(\dfrac{dr}{d\theta}\right)^2 \right\}^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

2.281 If $u = \dfrac{1}{r}$ the radius of curvature is

$$\rho = \frac{\left\{ u^2 + \left(\dfrac{du}{d\theta}\right)^2 \right\}^{\frac{3}{2}}}{u^3\left(u + \dfrac{d^2u}{d\theta^2}\right)}.$$

2.282 If the equation of the curve is given in the form,

$$r = f(s)$$

where s is the arc measured from a fixed point of the curve,

$$\rho = \frac{r\sqrt{1 - \left(\dfrac{dr}{ds}\right)^2}}{r\dfrac{d^2r}{ds^2} + \left(\dfrac{dr}{ds}\right)^2 - 1}.$$

2.283 If p is the perpendicular from the origin upon the tangent to the curve,

1. $$\rho = r\frac{dr}{dp}$$ 2. $$\rho = p + \frac{d^2p}{d\tau^2}$$

2.284 If $u = \dfrac{1}{r}$,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

2.285

$$\frac{d^2u}{d\theta^2} + u = \frac{r^2}{p^3}\left(\frac{dp}{dr}\right).$$

2.286 Polar coordinates of the center of curvature, r_1, θ_1:

$$r_1^2 = \frac{r^2\left\{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right\}^2 + \left(\dfrac{dr}{d\theta}\right)^2\left\{\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right\}^2}{\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right\}^2}$$

$$\theta_1 = \theta + \chi,$$

$$\tan\chi = \frac{\left(\dfrac{dr}{d\theta}\right)^3 + r^2\dfrac{dr}{d\theta}}{r\left(\dfrac{dr}{d\theta}\right)^2 - r^2\dfrac{d^2r}{d\theta^2}}.$$

2.287 If q is the chord of curvature (**2.225**):

$$q = 2p\frac{dr}{dp} = 2\rho\frac{p}{r},$$

$$= 2\frac{u^2 + \left(\dfrac{du}{d\theta}\right)^2}{u\left(u + \dfrac{d^2u}{d\theta^2}\right)}.$$

2.290 Rectilinear Asymptotes. If r approaches ∞ as θ approaches an angle α, and if $r(\alpha - \theta)$ approaches a limit, b, then the straight line

$$r\sin(\alpha - \theta) = b$$

is an asymptote to the curve $r = f(\theta)$.

2.295 Intrinsic Equation of a plane curve. An intrinsic equation of a plane curve is one giving the radius of curvature, ρ, as a function of the arc, s,

$$\rho = f(s)$$

If τ is the angle between the x-axis and the positive tangent (2.271):

$$d\tau = \frac{ds}{f(s)} \qquad\qquad x = x_0 + \int_{s_0}^{s} \cos\tau \cdot ds$$

$$\tau = \tau_0 + \int_{s_0}^{s} \frac{ds}{f(s)} \qquad\qquad y = y_0 + \int_{s_0}^{s} \sin\tau \cdot ds.$$

2.300 The general equation of the second degree:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$$

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}; \qquad a_{hk} = a_{kh}$$

A_{hk} = Minor of a_{hk}.

Criterion giving the nature of the curve:

<table>
<tr><td></td><td colspan="3">$A_{33} \neq 0$</td><td colspan="3">$A_{33} = 0$</td></tr>
<tr><td rowspan="4">$A \neq 0$</td><td>$A_{33}<0$</td><td colspan="2">$A_{33}>0$</td><td colspan="3" rowspan="4">Parabola</td></tr>
<tr><td rowspan="3">Hyperbola</td><td colspan="2">$a_{11}A$ or $a_{22}A$</td></tr>
<tr><td><0</td><td>>0</td></tr>
<tr><td>Ellipse</td><td>Imaginary Curve</td></tr>
<tr><td rowspan="4">$A = 0$</td><td rowspan="2">$A_{33}<0$</td><td colspan="2" rowspan="2">$A_{33}>0$</td><td colspan="2">A_{11} or A_{22}</td><td rowspan="2">$A_{11} = A_{22} = 0$</td></tr>
<tr><td><0</td><td>>0</td></tr>
<tr><td>Pair of Real Straight Lines</td><td colspan="2">Pair of Imaginary Lines</td><td>Real</td><td>Imaginary</td><td rowspan="2">Double Line</td></tr>
<tr><td colspan="3">Intersection Finite</td><td colspan="2">Pair of Parallel Lines</td></tr>
</table>

(Pascal: Repertorium der höheren Mathematik, II, 1, p. 238)

2.400 Parabola (Fig. 3).

2.401 O, Vertex; F, Focus; ordinate through D, Directrix.

Equation of parabola, origin at O,

$$y^2 = 4ax$$

$x = OM$, $y = MP$,

$OF = OD = a$

$FL = 2a =$ semi latus rectum.

$FP = D'P$.

2.402 $FP = FT = MD = x + a$.

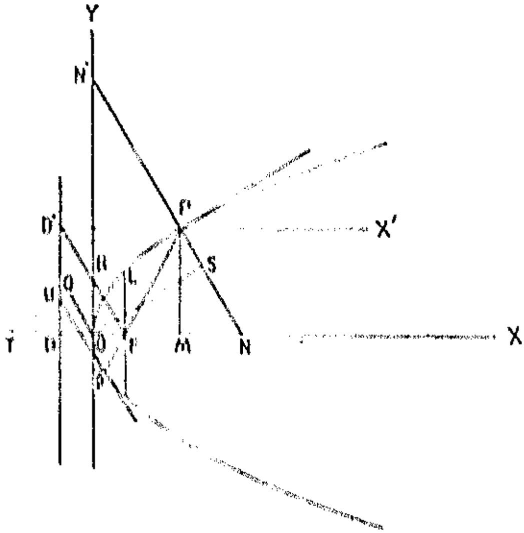

Fig. 3

$$NP = 2\sqrt{a(a+x)},\ TM = 2x,\ MN = 2a,\ ON = x + 2a.$$

$$ON' = \sqrt{\frac{x}{a}}\,(x+2a),\ OQ = x\sqrt{\frac{a}{a+x}},\ OS = (x+2a)\sqrt{\frac{x}{a+x}}.$$

FR perpendicular to tangent TP.

$$FR = \sqrt{a(a+x)},\ TP = 2TR = 2\sqrt{x(a+x)}.$$

$$\overline{FR}^2 = FT \times FO = FP \times FO.$$

The tangents TP and UP' at the extremities of a focal chord PFP' meet on the directrix at U at right angles.

$$\tau = \text{angle } XTP.$$

$$\tan \tau = \sqrt{\frac{a}{x}}.$$

The tangent at P bisects the angles FPD' and FUD'.

2.403 Radius of curvature:

$$\rho = \frac{2(x+a)^{\frac{3}{2}}}{\sqrt{a}} = \frac{1}{4}\,\frac{\overline{NP}^3}{a^2}.$$

Coördinates of center of curvature:

$$\xi = 3x + 2a,\ \eta = -2x\sqrt{\frac{x}{a}}.$$

Equation of Evolute:

2.404 Length of arc of parabola measured from vertex,

$$s = \sqrt{x(x+a)} + a \log\left(\sqrt{1+\frac{x}{a}} + \sqrt{\frac{x}{a}}\right).$$

Area $OPMO = \frac{1}{3}xy$.

2.405 Polar equation of parabola:

$$r = FP,$$
$$\theta = \text{angle } XFP,$$
$$r = \frac{2a}{1-\cos\theta}.$$

2.406 Equation of Parabola in terms of p, the perpendicular from F upon the tangent, and r, the radius vector FP:

$$\frac{l}{p^2} = \frac{2}{r}$$

l = semi latus rectum.

2.410 Ellipse (Fig. 4).

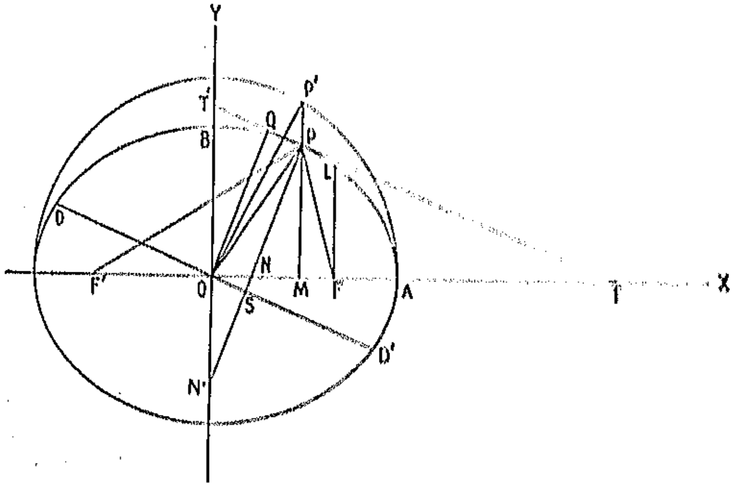

FIG. 4

2.411 O, Centre; F, F', Foci.

Equation of Ellipse origin at O:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2.412 Parametric Equations of Ellipse,

$$x = a \cos \phi, \qquad y = b \sin \phi.$$

ϕ = angle XOP', where P' is the point where the ordinate at P meets the eccentric circle, drawn with O as center and radius a.

2.413 $OF = OF' = ea$

$$e = \text{eccentricity} = \frac{\sqrt{a^2 - b^2}}{a},$$

$$FL = \frac{b^2}{a} = a(1 - e^2) = \text{semi latus rectum.}$$

$$F'P = a + ex, \; FP = a - ex, \; FP + F'P = 2a.$$

τ = angle XTT'.

$$\tan \tau = \frac{bx}{a\sqrt{a^2 - x^2}}.$$

$$NM = \frac{b^2x}{a^2}, \; ON = e^2x, \; OT = \frac{a^2}{x}, \; OT' = \frac{b^2}{y}, \; MT = \frac{a^2 - x^2}{x},$$

$$PT = \frac{\sqrt{a^2 - x^2}\sqrt{a^2 - e^2x^2}}{x}, \; ON' = \frac{e^2a}{b}\sqrt{a^2 - x^2}, \; PS = \frac{ab}{\sqrt{a^2 - e^2x^2}}$$

$$OS = \frac{e^2x\sqrt{a^2 - x^2}}{\sqrt{a^2 - e^2x^2}}.$$

2.414 DD' parallel to $T'T$; DD' and PP' are conjugate diameters:

$$OD^2 = a^2 - e^2x^2 = FP \times F'P.$$

$$OP^2 + OD^2 = a^2 + b^2.$$

$$PS \times OD = ab.$$

Equation of Ellipse referred to conjugate diameters as axes:

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \qquad \begin{array}{l} \alpha = \text{angle } XOP \\ \beta = \text{angle } XOD \end{array}$$

$$a' = OD' \qquad a'^2 = \frac{a^2b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \qquad \tan \alpha \tan \beta = -\frac{b^2}{a^2}$$

$$b' = OP \qquad b'^2 = \frac{a^2b^2}{a^2 \sin^2 \beta + b^2 \cos^2 \beta}$$

2.415 Radius of curvature of Ellipse:

$$\rho = \frac{(a^4y^2 + b^4x^2)^{\frac{3}{2}}}{a^4b^4} = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}.$$

$$\text{angle } FPN = \text{angle } F'PN = \omega,$$

$$\tan \omega = \frac{eay}{}$$

Coördinates of center of curvature:

$$\xi = \frac{e^2x^3}{a^2}, \ \eta = -\frac{a^2e^2y^3}{b^4}.$$

Equation of Evolute of Ellipse,

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} + \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1.$$

2.416 Area of Ellipse, πab.

Length of arc of Ellipse,

$$s = a\int_0^\phi \sqrt{1 - e^2 \sin^2 \phi}\, d\phi.$$

2.417 Polar Equation of Ellipse,

$$r = F'P, \ \theta = \text{angle } XF'P,$$

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

2.418

$$r = OP, \ \theta = \text{angle } XOP,$$

$$r = \frac{b}{\sqrt{1 - e^2 \cos^2 \theta}}$$

2.419 Equation of Ellipse in terms of p, the perpendicular from F upon the tangent at P, and r, the radius vector FP:

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a}.$$

l = semi latus rectum.

2.420 Hyperbola (Fig. 5).

2.421 O, Center; F, F', Foci.

Equation of hyperbola, origin at O,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x = OM, \ y = MP, \ a = OA = OA'.$$

2.422 Parametric Equations of hyperbola,

$$x = a \cosh u, \ y = b \sinh u.$$

or

$$x = a \sec \phi, \quad y = b \tan \phi.$$

ϕ = angle XOP', where P' is the point where the ordinate at T meets the circle of radius a, center O.

2.423 $OF = OF' = ea.$

$$e = \text{eccentricity} = \frac{\sqrt{a^2+b^2}}{a}.$$

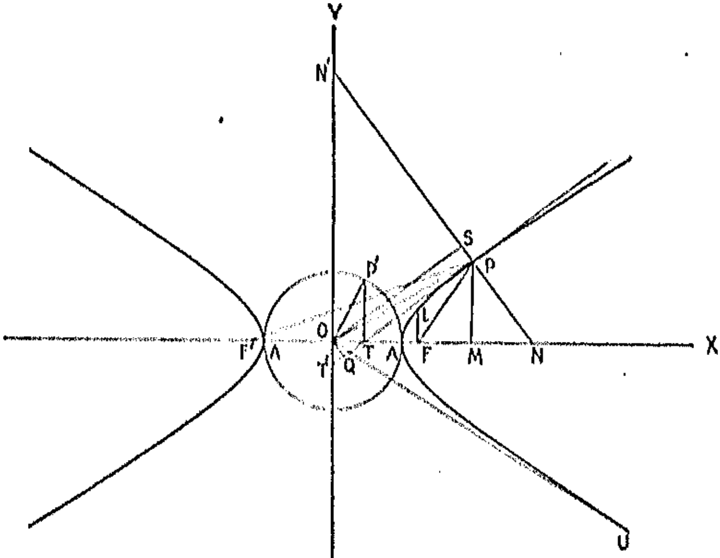

FIG. 5

$$FL = \frac{b^2}{a} = a(e^2-1) = \text{semi latus rectum.}$$

$$F'P = ex + a, \quad FP = ex - a, \quad F'P - FP = 2a.$$

$$\tau = \text{angle } XTP.$$

$$\tan \tau = \frac{bx}{a\sqrt{x^2-a^2}}.$$

$$NM = \frac{b^2x}{a^2}, \quad ON = e^2x, \quad OT = \frac{a^2}{x}, \quad OT' = \frac{b^2}{y},$$

$$MT = \frac{x^2-a^2}{x}, \quad PT = \frac{\sqrt{x^2-a^2}\sqrt{e^2x^2-a^2}}{x}, \quad ON' = \frac{e^2a}{b}\sqrt{x^2-a^2},$$

$$PS = \frac{ab}{\sqrt{e^2x^2-a^2}}, \quad OS = \frac{e^2x\sqrt{x^2-a^2}}{\sqrt{e^2x^2-a^2}}.$$

2.424 OU = Asymptote.

$$\tan XOU = \frac{b}{a}.$$

2.425 Radius of curvature of hyperbola,

$$\rho = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

$$\text{angle } F'PT = \text{angle } FPT.$$

$$\text{angle } FPN = \omega = \frac{\pi}{2} - FPT.$$

$$\text{angle } F'PN = \omega' = \frac{\pi}{2} + F'PT.$$

$$\tan\omega = \frac{aey}{b^2}.$$

$$\cos\omega = \frac{b}{\sqrt{e^2x^2 - a^2}}.$$

$$\frac{2}{\rho\cos\omega} = \frac{1}{FP} - \frac{1}{F'P}.$$

Coördinates of center of curvature,

$$\xi = \frac{e^2x^3}{a^2},\ \eta = -\frac{a^2e^2y^3}{b^4}.$$

Equation of Evolute of hyperbola,

$$\left(\frac{ax}{c^2}\right)^{\frac{2}{3}} - \left(\frac{by}{c^2}\right)^{\frac{2}{3}} = 1.$$

2.426 In a rectangular hyperbola $b = a$; the asymptotes are perpendicular to each other. Equation of rectangular hyperbola with asymptotes as axes and origin at O:

$$xy = \frac{a^2}{2}.$$

2.427 Length of arc of hyperbola,

$$s = \frac{b^2}{ae}\int_0^{\phi}\frac{\sec^2\phi\, d\phi}{\sqrt{1 - k^2\sin^2\phi}},\quad k = \frac{1}{e},\quad \tan\phi = \frac{aey}{b^2}.$$

2.428 Polar Equation of hyperbola:

$$r = F'P,\quad \theta = XF'P,\quad r = a\frac{e^2 - 1}{e\cos\theta - 1}.$$

$$r = OP,\quad \theta = XOP,\quad r^2 = \frac{b^2}{e^2\cos^2\theta - 1}.$$

2.429 Equation of right-hand branch of hyperbola in terms of p, the perpendicular from F upon the tangent at P and r, the radius vector FP.

$$\frac{l}{p^2} = \frac{2}{r} + \frac{1}{a}.$$

$$l = \text{semi latus rectum}.$$

2.450 Cycloids and Trochoids.

If a circle of radius a rolls on a straight line as base the extremity of any radius, a, describes a cycloid. The rectangular equation of a cycloid is:

$$x = a(\phi - \sin\phi),$$
$$y = a(1 - \cos\phi),$$

where the x-axis is the base with the origin at the initial point of contact. ϕ is the angle turned through by the moving circle. (Fig. 6.)

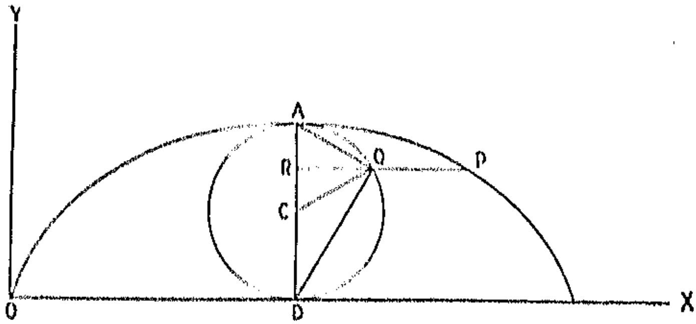

FIG. 6

A = vertex of cycloid,

C = center of generating circle, drawn tangent at A.

The tangent to the cycloid at P is parallel to the chord AQ.

$$\text{Arc } AP = 2 \times \text{chord } AQ.$$

The radius of curvature at P is parallel to the chord QD and equal to $2 \times$ chord QD.

$$PQ = \text{circular arc } AQ.$$

$$\text{Length of cycloid: } s = 8a;\ a = CA.$$

$$\text{Area of cycloid: } S = 3\pi a^2.$$

2.451 A point on the radius, $b > a$, describes a prolate trochoid. A point, $b < a$, describes a curtate trochoid. The general equation of trochoids and cycloids is

$$x = a\phi - (a + d)\sin\phi,$$
$$y = (a + d)(1 - \cos\phi),$$
$$d = 0 \text{ Cycloid},$$
$$d > 0 \text{ Prolate trochoid},$$
$$d < 0 \text{ Curtate trochoid}.$$

Radius of curvature:

$$\rho = \frac{(2ay + d^2)^{3/2}}{ay + ad + d^2}.$$

2.452 Epi- and Hypocycloids. An epicycloid is described by a point on a circle of radius a that rolls on the convex side of a fixed circle of radius b. An hypocycloid is described by a point on a circle of radius a that rolls on the concave side of a fixed circle of radius b.

Equations of epi- and hypocycloids.

Upper sign: Epicycloid,
Lower sign: Hypocycloid.

$$x = (b \pm a) \cos \phi \mp a \cos \frac{b \pm a}{a} \phi,$$

$$y = (b \pm a) \sin \phi - a \sin \frac{b \pm a}{a} \phi.$$

The origin is at the center of the fixed circle. The x axis is the line joining the centers of the two circles in the initial position and ϕ is the angle turned through by the moving circle.

Radius of curvature:

$$\rho = \frac{2a(b \pm a)}{b \pm 2a} \sin \frac{a}{2b} \phi.$$

2.453 In the epicycloid put $b = a$. The curve becomes a Cardioid:

$$(x^2 + y^2)^2 - 6a^2(x^2 + y^2) + 8a^3x = 3a^4.$$

2.454 Catenary. The equation may be written:

1. $$y = \frac{1}{2} a(e^{\frac{x}{a}} + e^{-\frac{x}{a}}).$$

2. $$y = a \cosh \frac{x}{a}.$$

3. $$x = a \log \frac{y \pm \sqrt{y^2 - a^2}}{a}.$$

The radius of curvature, which is equal to the length of the normal, is:

$$\rho = a \cosh^2 \frac{x}{a}.$$

2.455 Spiral of Archimedes. A point moving uniformly along a line which rotates uniformly about a fixed point describes a spiral of Archimedes. The equation is:

$$r = a\theta,$$

or

$$\sqrt{x^2 + y^2} = a \tan^{-1} \frac{y}{x}.$$

The polar subtangent = polar subnormal = a.

Radius of curvature:

$$\rho = \frac{r(1 + \theta^2)^{\frac{3}{2}}}{\theta(2 + \theta^2)} = \frac{(r^2 + a^2)^{\frac{3}{2}}}{r^2 + 2a^2}.$$

2.456 Hyperbolic spiral:

$$r\theta = a.$$

2.457 Parabolic spiral:

$$r^2 = a^2\theta.$$

2.458 Logarithmic or equiangular spiral:

$r = ae^{n\theta}$,
$n = \cot \alpha = \text{const.}$,
$\alpha =$ angle tangent to curve makes with the radius vector.

2.459 Lituus:

$$r\sqrt{\theta} = a.$$

2.460 Neoid:

$$r = a + b\theta.$$

2.461 Cissoid:

$$(x^2 + y^2)x = 2ay^2,$$
$$r = 2a \tan \theta \sin \theta.$$

2.462 Cassinoid:

$$(x^2 + y^2 + a^2)^2 = 4a^2x^2 + b^4,$$
$$r^4 - 2a^2r^2 \cos 2\theta = b^4 - a^4.$$

2.463 Lemniscate ($b = a$ in Cassinoid):

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2),$$
$$r^2 = 2a^2 \cos 2\theta.$$

2.464 Conchoid:

$$x^2y^2 = (b + y)^2(a^2 - y^2).$$

2.465 Witch of Agnesi:

$$x^2y = 4a^2(2a - y).$$

2.466 Tractrix:

$$x = \tfrac{1}{2}a \log \frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}} - \sqrt{a^2 - y^2},$$

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}},$$

$$\rho = \frac{a\sqrt{a^2 - y^2}}{y}.$$

SOLID GEOMETRY

2.600 The Plane. The general equation of the plane is:

$$Ax + By + Cz + D = 0.$$

2.601 l, m, n are the direction cosines of the normal to the plane and p is the perpendicular distance from the origin upon the plane.

$$l,\ m,\ n = \frac{A,\ B,\ C}{\sqrt{A^2 + B^2 + C^2}},$$

$$p = lx + my + nz,$$

$$p = -\frac{D}{\sqrt{A^2 + B^2 + C^2}}.$$

2.602 The perpendicular from the point x_1, y_1, z_1 upon the plane $Ax + By + Cz + D = 0$ is:

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

2.603 θ is the angle between the two planes:

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0,$$

$$\cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

2.604 Equation of the plane passing through the three points (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3):

$$x\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} + y\begin{vmatrix} z_1 & x_1 & 1 \\ z_2 & x_2 & 1 \\ z_3 & x_3 & 1 \end{vmatrix} + z\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

THE RIGHT LINE

2.620 The equations of a right line passing through the point x_1, y_1, z_1, and whose direction cosines are l, m, n are:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

2.621 θ is the angle between the two lines whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2:

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2,$$
$$\sin^2\theta = (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2.$$

2.622 The direction cosines of the normal to the plane defined by the two lines whose direction cosines are l_1, m_1, n_1 and $l_2, m_2 n_2$ are:

$$\frac{m_1n_2 - m_2n_1}{\sin\theta}, \quad \frac{n_1l_2 - n_2l_1}{\sin\theta}, \quad \frac{l_1m_2 - l_2m_1}{\sin\theta}.$$

2.623 The shortest distance between the two lines:

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \quad \text{and} \quad \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2},$$

is:

$$d = \frac{(x_1 - x_2)(m_1n_2 - m_2n_1) + (y_1 - y_2)(n_1l_2 - n_2l_1) + (z_1 - z_2)(l_1m_2 - l_2m_1)}{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}^{\frac{1}{2}}}.$$

2.624 The direction cosines of the shortest distance between the two lines are:

$$\frac{(m_1n_2 - n_2m_1),\ (n_1l_2 - n_2l_1),\ (l_1m_2 - l_2m_1)}{\{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2\}^{\frac{1}{2}}}.$$

2.625 The perpendicular distance from the point x_2, y_2, z_2 to the line:

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$$

is:

$$d = \{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}^{\frac{1}{2}} - \{l_1(x_2-x_1)+m_1(y_2-y_1)+n_1(z_2-z_1)\}.$$

2.626 The direction cosines of the line passing through the two points x_1, y_1, z_1 and x_2, y_2, z_2 are:

$$\frac{(x_2-x_1),\quad (y_2-y_1),\quad (z_2-z_1)}{\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\}^{\frac{1}{2}}}.$$

2.627 The two lines:

$$\begin{aligned} x &= m_1z+p_1, \\ y &= n_1z+q_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= m_2z+p_2, \\ y &= n_2z+q_2 \end{aligned}$$

intersect at a point if,

$$(m_1-m_2)(q_1-q_2)-(n_1-n_2)(p_1-p_2)=0.$$

The coördinates of the point of intersection are:

$$x=\frac{m_1p_2-m_2p_1}{m_1-m_2},\quad y=\frac{n_1q_2-n_2q_1}{n_1-n_2},\quad z=\frac{p_2-p_1}{m_1-m_2}=\frac{q_2-q_1}{n_1-n_2}.$$

The equation of the plane containing the two lines is then

$$(n_1-n_2)(x-m_1z-p_1)=(m_1-m_2)(y-n_1z-q_1).$$

SURFACES

2.640 A single equation in x, y, z represents a surface:

$$F(x,y,z)=0.$$

2.641 The direction cosines of the normal to the surface are:

$$l,\ m,\ n=\frac{\dfrac{\partial F}{\partial x},\quad \dfrac{\partial F}{\partial y},\quad \dfrac{\partial F}{\partial z}}{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2+\left(\dfrac{\partial F}{\partial y}\right)^2+\left(\dfrac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}}}.$$

2.642 The perpendicular from the origin upon the tangent plane at x, y, z is:

$$p=lx+my+nz.$$

2.643 The two principal radii of curvature of the surface $F(x,y,z)=0$ are given by the two roots of:

$$\begin{vmatrix} \frac{k}{\rho} + \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{k}{\rho} + \frac{\partial^2 F}{\partial y^2} & \frac{\partial^2 F}{\partial y \partial z} & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial x \partial z} & \frac{\partial^2 F}{\partial y \partial z} & \frac{k}{\rho} + \frac{\partial^2 F}{\partial z^2} & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & 0 \end{vmatrix} = 0,$$

where:

$$k^2 = \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2.$$

2.644 The coördinates of each center of curvature are:

$$\xi = x + \frac{\rho}{k}\frac{\partial F}{\partial x}, \qquad \eta = y + \frac{\rho}{k}\frac{\partial F}{\partial y}, \qquad \zeta = z + \frac{\rho}{k}\frac{\partial F}{\partial z}.$$

2.645 The envelope of a family of surfaces:

1. $$F(x, y, z, \alpha) = 0$$

is found by eliminating α between (1) and

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

2.646 The characteristic of a surface is a curve defined by the two equations (1) and (2) in **2.645.**

2.647 The envelope of a family of surfaces with two variable parameters, α, β, is obtained by eliminating α and β between:

1. $$F(x, y, z, \alpha, \beta) = 0.$$

2. $$\frac{\partial F}{\partial \alpha} = 0.$$

3. $$\frac{\partial F}{\partial \beta} = 0.$$

2.648 The equations of a surface may be given in the parametric form:

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v).$$

The equation of a tangent plane at x_1, y_1, z_1 is:

$$(x - x_1)\frac{\partial(f_2, f_3)}{\partial(u, v)} + (y - y_1)\frac{\partial(f_3, f_1)}{\partial(u, v)} + (z - z_1)\frac{\partial(f_1, f_2)}{\partial(u, v)} = 0,$$

where

$$\frac{\partial(f_2, f_3)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{vmatrix}, \text{ etc.}$$ See **1.370.**

2.649 The direction cosines to the normal to the surface in the form **2.648** are:

$$l,\ m,\ n = \frac{\dfrac{\partial(f_2, f_3)}{\partial(u, v)},\ \dfrac{\partial(f_3, f_1)}{\partial(u, v)},\ \dfrac{\partial(f_1, f_2)}{\partial(u, v)}}{\left\{\left(\dfrac{\partial(f_2, f_3)}{\partial(u, v)}\right)^2 + \left(\dfrac{\partial(f_3, f_1)}{\partial(u, v)}\right)^2 + \left(\dfrac{\partial(f_1, f_2)}{\partial(u, v)}\right)^2\right\}^{\frac{1}{2}}}.$$

2.650 If the equation of the surface is:

$$z = f(x, y),$$

the equation of the tangent plane at x_1, y_1, z_1 is:

$$z - z_1 = \left(\frac{\partial f}{\partial x}\right)_1 (x - x_1) + \left(\frac{\partial f}{\partial y}\right)_1 (y - y_1).$$

2.651 The direction cosines of the normal to the surface in the form **2.650** are:

$$l,\ m,\ n = \frac{-\left(\dfrac{\partial f}{\partial x}\right),\ -\left(\dfrac{\partial f}{\partial y}\right),\ +1}{\left\{1 + \left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2\right\}^{\frac{1}{2}}}.$$

2.652 The two principal radii of curvature of the surface in the form **2.650** are given by the two roots of:

$$(rt - s^2)\rho^2 - \{(1 + q^2)r - 2pqs + (1 + p^2)t\}\sqrt{1 + p^2 + q^2}\,\rho + (1 + p^2 + q^2)^2 = 0,$$

where

$$p = \frac{\partial f}{\partial x},\quad q = \frac{\partial f}{\partial y},\quad r = \frac{\partial^2 f}{\partial x^2},\quad s = \frac{\partial^2 f}{\partial x \partial y},\quad t = \frac{\partial^2 f}{\partial y^2}.$$

2.653 If ρ_1 and ρ_2 are the two principal radii of curvature of a surface, and ρ is the radius of curvature in a plane making an angle ϕ with the plane of ρ_1,

$$\frac{1}{\rho} = \frac{\cos^2 \phi}{\rho_1} + \frac{\sin^2 \phi}{\rho_2}.$$

2.654 If ρ and ρ' are the radii of curvature in any two mutually perpendicular planes, and ρ_1 and ρ_2 the two principal radii of curvature:

$$\frac{1}{\rho} + \frac{1}{\rho'} = \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

2.655 Gauss's measure of the curvature of a surface is:

$$\frac{1}{\rho} = \frac{1}{\rho_1 \rho_2}.$$

SPACE CURVES

2.670 The equations of a space curve may be given in the forms:

(a) $\qquad F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0.$

(b) $\qquad x = f_1(t), \quad y = f_2(t), \quad z = f_3(t).$

(c) $\qquad y = \phi(x), \quad z = \psi(x).$

2.671 The direction cosines of the tangent to a space curve in the form (a) are:

$$l = \frac{\dfrac{\partial F_1}{\partial y}\dfrac{\partial F_2}{\partial z} - \dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial y}}{T},$$

$$m = \frac{\dfrac{\partial F_1}{\partial z}\dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial x}\dfrac{\partial F_2}{\partial z}}{T},$$

$$n = \frac{\dfrac{\partial F_1}{\partial x}\dfrac{\partial F_2}{\partial y} - \dfrac{\partial F_1}{\partial y}\dfrac{\partial F_2}{\partial x}}{T},$$

where T is the positive root of:

$$T^2 = \left\{\left(\frac{\partial F_1}{\partial x}\right)^2 + \left(\frac{\partial F_1}{\partial y}\right)^2 + \left(\frac{\partial F_1}{\partial z}\right)^2\right\}\left\{\left(\frac{\partial F_2}{\partial x}\right)^2 + \left(\frac{\partial F_2}{\partial y}\right)^2 + \left(\frac{\partial F_2}{\partial z}\right)^2\right\}$$
$$- \left\{\frac{\partial F_1}{\partial x}\frac{\partial F_2}{\partial x} + \frac{\partial F_1}{\partial y}\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}\right\}^2.$$

2.672 The direction cosines of the tangent to a space curve in the form (b) are:

$$l,\ m,\ n = \frac{x',\ y',\ z'}{[x'^2 + y'^2 + z'^2]^{\frac{1}{2}}},$$

where the accents denote differentials with respect to t.

2.673 If s, the length of arc measured from a fixed point on the curve is the parameter, t:

$$l,\ m,\ n = \frac{dx}{ds},\ \frac{dy}{ds},\ \frac{dz}{ds}.$$

2.674 The principal radius of curvature of a space curve in the form (b) is:

$$\rho = \frac{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}}{[(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2]^{\frac{1}{2}}}$$

$$= \frac{s'^2}{(x''^2 + y''^2 + z''^2 - s''^2)^{\frac{1}{2}}},$$

where the double accents denote second differentials with respect to t, and s, the length of arc, is a function of t.

2.675 When $t = s$:

$$\frac{1}{\rho} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}}$$

2.676 The direction cosines of the principal normal to the space curve in the form (b) are:

$$l' = \frac{z'(z'x'' - x'z'') - y'(x'y'' - y'x'')}{L},$$

$$m' = \frac{x'(x'y'' - y'x'') - z'(y'z'' - z'y'')}{L},$$

$$n' = \frac{y'(y'z'' - z'y'') - x'(z'x'' - x'z'')}{L},$$

where

$$L = \{x'^2 + y'^2 + z'^2\}^{\frac{1}{2}}\{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\}^{\frac{1}{2}}.$$

2.677 The direction cosines of the binormal to the curve in the form (b) are:

$$l'' = \frac{y'z'' - z'y''}{S},$$

$$m'' = \frac{z'x'' - x'z''}{S},$$

$$n'' = \frac{x'y'' - y'x''}{S},$$

where

$$S = \{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2\}^{\frac{1}{2}}.$$

2.678 If s, the distance measured along the curve from a fixed point on it is the parameter, t:

$$l' = \rho\frac{d^2x}{ds^2},\ m' = \rho\frac{d^2y}{ds^2},\ n' = \rho\frac{d^2z}{ds^2},$$

where ρ is the principal radius of curvature; and

$$l'' = \rho\left(\frac{dy}{ds}\frac{d^2z}{ds^2} - \frac{dz}{ds}\frac{d^2y}{ds^2}\right),$$

$$m'' = \rho\left(\frac{dz}{ds}\frac{d^2x}{ds^2} - \frac{dx}{ds}\frac{d^2z}{ds^2}\right),$$

$$n'' = \rho\left(\frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}\right).$$

2.679 The radius of torsion, or radius of second curvature of a space curve is:

$$T = \frac{(x'^2 + y'^2 + z'^2)^{\frac{1}{2}}}{\left\{\left(\frac{\partial l''}{\partial t}\right)^2 + \left(\frac{\partial m''}{\partial t}\right)^2 + \left(\frac{\partial n''}{\partial t}\right)^2\right\}^{\frac{1}{2}}}$$

$$= -\frac{1}{S^2}\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

where S is given in **2.677**.

2.680 When $t = s$:

$$\frac{1}{T} = \left\{\left(\frac{\partial l''}{\partial s}\right)^2 + \left(\frac{\partial m''}{\partial s}\right)^2 + \left(\frac{\partial n''}{\partial s}\right)^2\right\}^{\frac{1}{2}}$$

$$= -\rho^2 \begin{vmatrix} \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{d^3x}{ds^3} & \frac{d^3y}{ds^3} & \frac{d^3z}{ds^3} \end{vmatrix}.$$

2.681 The direction cosines of the tangent to a space curve in the form (c) are:

$$l,\ m,\ n = \frac{1,\ y',\ z'}{\sqrt{1 + y'^2 + z'^2}}$$

where accents denote differentials with respect to x:

$$y' = \frac{d\phi(x)}{dx}, \quad z' = \frac{d\psi(x)}{dx}.$$

2.682 The principal radius of curvature of a space curve in the form (c) is:

$$\rho = \left\{ \frac{(1 + y'^2 + z'^2)^3}{(y'z'' - z'y'')^2 + y''^2 + z''^2} \right\}^{\frac{1}{2}}.$$

2.683 The radius of torsion of a space curve in the form (c) is:

$$\tau = \frac{(1 + y'^2 + z'^2)^3}{\rho^2(y''z''' - z''y''')}.$$

2.690 The relation between the direction cosines of the tangent, principal normal and binormal to a space curve is:

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 1.$$

2.691 The tangent, principal normal and binormal all being mutually perpendicular the relations of **2.00** hold among their direction cosines.

III. TRIGONOMETRY

3.00 $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\csc x = \dfrac{1}{\sin x}$, $\cot x = \dfrac{1}{\tan x}$,

$\sec^2 x = 1 + \tan^2 x$, $\csc^2 x = 1 + \cot^2 x$, $\sin^2 x + \cos^2 x = 1$,

$\operatorname{versin} x = 1 - \cos x$, $\operatorname{coversin} x = 1 - \sin x$, $\operatorname{haversin} x = \sin^2 \dfrac{x}{2}$.

3.01 $$\sin x = -\sin(-x) = \sqrt{\frac{1 - \cos 2x}{2}} = 2\sqrt{\cos^2\frac{x}{2} - \cos^4\frac{x}{2}},$$

$$= 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}},$$

$$= \frac{1}{\sqrt{1 + \cot^2 x}} = \frac{1}{\cot\frac{x}{2} - \cot x} = \frac{1}{\tan\frac{x}{2} + \cot x},$$

$$= \cot\frac{x}{2}\cdot(1 - \cos x) = \tan\frac{x}{2}\cdot(1 + \cos x),$$

$$= \sin y \cos(x - y) + \cos y \sin(x - y),$$

$$= \cos y \sin(x + y) - \sin y \cos(x + y),$$

$$= -\tfrac{1}{2} i (e^{ix} - e^{-ix}).$$

3.02 $$\cos x = \cos(-x) = \sqrt{\frac{1 + \cos 2x}{2}} = 1 - 2\sin^2\frac{x}{2},$$

$$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = \frac{1}{\sqrt{1 + \tan^2 x}},$$

$$= \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} = \frac{1}{1 + \tan x \tan\frac{x}{2}} = \frac{1}{\tan x \cot\frac{x}{2} - 1},$$

$$= \frac{\cot\frac{x}{2} - \tan\frac{x}{2}}{\cot\frac{x}{2} + \tan\frac{x}{2}} = \frac{\cot x}{\sqrt{1 + \cot^2 x}} = \frac{\sin 2x}{2\sin x},$$

$$= \cos y \cos(x + y) + \sin y \sin(x + y),$$

$$= \cos y \cos(x - y) - \sin y \sin(x - y),$$

$$= \tfrac{1}{2}(e^{ix} + e^{-ix}).$$

3.03 $\tan x = -\tan(-x) = \dfrac{\sin 2x}{1+\cos 2x} = \dfrac{1-\cos 2x}{\sin 2x}$,

$$= \sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \frac{\sin(x+y)+\sin(x-y)}{\cos(x+y)+\cos(x-y)},$$

$$= \frac{\cos(x-y)-\cos(x+y)}{\sin(x+y)-\sin(x-y)} = \cot x - 2\cot 2x,$$

$$= \frac{\tan\frac{x}{2}}{1-\tan\frac{x}{2}} + \frac{\tan\frac{x}{2}}{1+\tan\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}},$$

$$= \frac{1}{1-\tan\frac{x}{2}} - \frac{1}{1+\tan\frac{x}{2}},$$

$$= i\,\frac{1-e^{2ix}}{1+e^{2ix}}.$$

3.04 The values of five trigonometric functions in terms of the sixth are given in the following table. (For signs, see **3.05**.)

	$\sin x = a$	$\cos x = a$	$\tan x = a$	$\cot x = a$	$\sec x = a$	$\csc x = a$
$\sin x =$	a	$\sqrt{1-a^2}$	$\dfrac{a}{\sqrt{1+a^2}}$	$\dfrac{1}{\sqrt{1+a^2}}$	$\dfrac{\sqrt{a^2-1}}{a}$	$\dfrac{1}{a}$
$\cos x =$	$\sqrt{1-a^2}$	a	$\dfrac{1}{\sqrt{1+a^2}}$	$\dfrac{a}{\sqrt{1+a^2}}$	$\dfrac{1}{a}$	$\dfrac{\sqrt{a^2-1}}{a}$
$\tan x =$	$\dfrac{a}{\sqrt{1-a^2}}$	$\dfrac{\sqrt{1-a^2}}{a}$	a	$\dfrac{1}{a}$	$\sqrt{a^2-1}$	$\dfrac{1}{\sqrt{a^2-1}}$
$\cot x =$	$\dfrac{\sqrt{1-a^2}}{a}$	$\dfrac{a}{\sqrt{1-a^2}}$	$\dfrac{1}{a}$	a	$\dfrac{1}{\sqrt{a^2-1}}$	$\sqrt{a^2-1}$
$\sec x =$	$\dfrac{1}{\sqrt{1-a^2}}$	$\dfrac{1}{a}$	$\sqrt{1+a^2}$	$\dfrac{\sqrt{1+a^2}}{a}$	a	$\dfrac{a}{\sqrt{a^2-1}}$
$\csc x =$	$\dfrac{1}{a}$	$\dfrac{1}{\sqrt{1-a^2}}$	$\dfrac{\sqrt{1+a^2}}{a}$	$\sqrt{1+a^2}$	$\dfrac{a}{\sqrt{a^2-1}}$	a

3.05 The trigonometric functions are periodic, the periods of the sin, cos, sec, csc being 2π, and those of the tan and cot, π. Their signs may be determined from the following table. In using formulas giving any of the trigonometric

functions by the root of some quantity, the proper sign may be taken from this table.

	$0°$	$0-\frac{\pi}{2}$ $0-90°$	$\frac{\pi}{2}$ $90°$	$\frac{\pi}{2}-\pi$ $90°-180°$	π $180°$	$\pi-\frac{3}{2}\pi$ $180°-270°$	$\frac{3}{2}\pi$ $270°$	$\frac{3}{2}\pi-2\pi$ $270°-360°$	2π $360°$
sin	0	+	1	+	0	−	−1	−	0
cos	1	+	0	−	−1	−	0	+	1
tan	0	+	$\pm\infty$	−	0	+	$\pm\infty$	−	0
cot	$\pm\infty$	+	0	−	$\pm\infty$	+	0	−	$\mp\infty$
sec	1	+	$\pm\infty$	−	−1	−	$\pm\infty$	+	1
csc	$\pm\infty$	+	1	+	$\pm\infty$	−	−1	−	$\mp\infty$

3.10 Functions of Half an Angle. (See **3.05** for signs.)

3.101
$$\sin\tfrac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{2}},$$
$$= \tfrac{1}{2}\left\{\pm\sqrt{1+\sin x} \mp \sqrt{1-\sin x}\right\}$$
$$= \pm\sqrt{\tfrac{1}{2}\left(1 - \frac{1}{\pm\sqrt{1+\tan^2 x}}\right)}$$

3.102
$$\cos\tfrac{1}{2}x = \pm\sqrt{\frac{1+\cos x}{2}},$$
$$= \tfrac{1}{2}\left\{\pm\sqrt{1+\sin x} \pm \sqrt{1-\sin x}\right\},$$
$$= \pm\sqrt{\tfrac{1}{2}\left(1 + \frac{1}{\pm\sqrt{1+\tan^2 x}}\right)}.$$

3.103
$$\tan\tfrac{1}{2}x = \pm\sqrt{\frac{1-\cos x}{1+\cos x}},$$

$$= \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x},$$

$$= \frac{\pm\sqrt{1 + \tan^2 x} - 1}{\tan x}.$$

3.11 Functions of the Sum and Difference of Two Angles.

3.111 $$\sin (x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$= \cos x \cos y (\tan x \pm \tan y),$$
$$= \frac{\tan x \pm \tan y}{\tan x \mp \tan y} \sin (x \mp y),$$
$$= \frac{1}{2}\left\{ \cos (x + y) + \cos (x - y) \right\} (\tan x \pm \tan y).$$

3.112 $$\cos (x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$= \cos x \cos y (1 \mp \tan x \tan y),$$
$$= \frac{\cot x \mp \tan y}{\cot x \pm \tan y} \cos (x \mp y),$$
$$= \frac{\cot y \mp \tan x}{\cot y \tan x \mp 1} \sin (x \mp y),$$
$$= \cos x \sin y (\cot y \mp \tan x).$$

3.113 $$\tan (x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$= \frac{\cot y \pm \cot x}{\cot x \cot y \mp 1},$$
$$= \frac{\sin 2x \pm \sin 2y}{\cos 2x + \cos 2y}.$$

3.114 $$\cot (x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x},$$
$$= -\frac{\sin 2x \mp \sin 2y}{\cos 2x - \cos 2y}.$$

3.115 The cosine and sine of the sum of any number of angles in terms of the sine and cosine of the angles are given by the real and imaginary parts of

$$\cos (x_1 + x_2 + \dots + x_n) + i \sin (x_1 + x_2 + \dots + x_n)$$
$$= (\cos x_1 + i \sin x_1)(\cos x_2 + i \sin x_2) \dots (\cos x_n + i \sin x_n)$$

3.12 Sums and Differences of Trigonometric Functions.

3.121 $$\sin x \pm \sin y = 2 \sin \tfrac{1}{2}(x \pm y) \cos \tfrac{1}{2}(x \mp y),$$
$$= (\cos x + \cos y) \tan \tfrac{1}{2}(x \pm y),$$
$$= (\cos y - \cos x) \cot \tfrac{1}{2}(x \mp y),$$
$$= \frac{\tan \tfrac{1}{2}(x \pm y)}{\tan \tfrac{1}{2}(x \mp y)} (\sin x \mp \sin y).$$

3.122 $$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y),$$
$$= \frac{\sin x \pm \sin y}{\tan \tfrac{1}{2}(x \pm y)},$$
$$= \frac{\cot \tfrac{1}{2}(x + y)}{\tan \tfrac{1}{2}(x - y)} (\cos y - \cos x).$$

3.123 $$\cos x - \cos y = 2 \sin \tfrac{1}{2}(y + x) \sin \tfrac{1}{2}(y - x)$$
$$= -(\sin x \pm \sin y) \tan \tfrac{1}{2}(x \mp y).$$

3.124 $$\tan x \pm \tan y = \frac{\sin (x \pm y)}{\cos x \cdot \cos y},$$
$$= \frac{\sin (x \pm y)}{\sin (x \mp y)} (\tan x \mp \tan y),$$
$$= \tan y \tan (x \pm y)(\cot y \mp \tan x),$$
$$= \frac{1 \mp \tan x \tan y}{\cot (x \pm y)},$$
$$= (1 \mp \tan x \tan y) \tan (x \pm y).$$

3.125 $$\cot x \pm \cot y = \pm \frac{\sin (x \pm y)}{\sin x \sin y}.$$

3.130

1. $$\frac{\sin x \pm \sin y}{\cos x + \cos y} = \tan \tfrac{1}{2}(x \pm y).$$

2. $$\frac{\sin x \pm \sin y}{\cos x - \cos y} = -\cot \tfrac{1}{2}(x \mp y).$$

3. $$\frac{\sin x + \sin y}{\sin x - \sin y} = \frac{\tan \tfrac{1}{2}(x + y)}{\tan \tfrac{1}{2}(x - y)}.$$

3.140

1. $$\sin^2 x + \sin^2 y = 1 - \cos(x+y)\cos(x-y).$$
2. $$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$$
$$= \sin(x+y)\sin(x-y).$$
3. $$\cos^2 x - \sin^2 y = \cos(x+y)\cos(x-y).$$
4. $$\sin^2(x+y) + \sin^2(x-y) = 1 - \cos 2x \cos 2y.$$
5. $$\sin^2(x+y) - \sin^2(x-y) = \sin 2x \sin 2y.$$
6. $$\cos^2(x+y) + \cos^2(x-y) = 1 + \cos 2x \cos 2y.$$
7. $$\cos^2(x+y) - \cos^2(x-y) = -\sin 2x \sin 2y.$$

3.150

1. $$\cos nx \cos mx = \tfrac{1}{2}\cos(n-m)x + \tfrac{1}{2}\cos(n+m)x.$$
2. $$\sin nx \sin mx = \tfrac{1}{2}\cos(n-m)x - \tfrac{1}{2}\cos(n+m)x.$$
3. $$\cos nx \sin mx = \tfrac{1}{2}\sin(n+m)x - \tfrac{1}{2}\sin(n-m)x.$$

3.160

1. $$e^{x+iy} = e^x(\cos y + i\sin y).$$
2. $$a^{x+iy} = a^x\{\cos(y\log a) + i\sin(y\log a)\}.$$
3. $$(\cos x \pm i\sin x)^n = \cos nx \pm i\sin nx$$
[De Moivre's Theorem].
4. $$\sin(x \pm iy) = \sin x\cosh y \pm i\cos x\sinh y.$$
5. $$\cos(x \pm iy) = \cos x\cosh y \mp i\sin x\sinh y.$$
6. $$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}).$$
7. $$\sin x = -\frac{i}{2}(e^{ix} - e^{-ix}).$$
8. $$e^{ix} = \cos x + i\sin x.$$
9. $$e^{-ix} = \cos x - i\sin x.$$

3.170 Sines and Cosines of Multiple Angles.

3.171 n an even integer:

$$\sin nx = n\cos x\left\{\sin x - \frac{(n^2-2^2)}{3!}\sin^3 x + \frac{(n^2-2^2)(n^2-4^2)}{5!}\sin^5 x - \dots\right\}.$$

$$\cos nx = 1 - \frac{n^2}{2!}\sin^2 x + \frac{n^2(n^2-2^2)}{4!}\sin^4 x - \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}\sin^6 x + \dots$$

3.172 n an odd integer:

$$\sin nx = n\left\{\sin x - \frac{(n^2-1^2)}{3!}\sin^3 x + \frac{(n^2-1^2)(n^2-3^2)}{5!}\sin^5 x - \ldots\right\}.$$

$$\cos nx = \cos x\left\{1 - \frac{(n^2-1^2)}{2!}\sin^2 x + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4 x - \ldots\right\}.$$

3.173 n an even integer:

$$\sin nx = (-1)^{\frac{n}{2}+1}\cos x\left\{2^{n-1}\sin^{n-1}x - \frac{(n-2)}{1!}2^{n-3}\sin^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\sin^{n-7}x + \ldots\right\}.$$

$$\cos nx = (-1)^{\frac{n}{2}}\left\{2^{n-1}\sin^n x - \frac{n}{1!}2^{n-3}\sin^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x - \frac{n(n-3)(n-5)}{3!}2^{n-7}\sin^{n-6}x + \ldots\right\}.$$

3.174 n an odd integer:

$$\sin nx = (-1)^{\frac{n-1}{2}}\left\{2^{n-1}\sin^n x - \frac{n}{1!}2^{n-3}\sin^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\sin^{n-4}x - \frac{n(n-3)(n-5)}{3!}2^{n-7}\sin^{n-6}x + \ldots\right\}.$$

$$\cos nx = (-1)^{\frac{n-1}{2}}\cos x\left\{2^{n-1}\sin^{n-1}x - \frac{n-2}{1!}2^{n-3}\sin^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\sin^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\sin^{n-7}x + \ldots\right\}.$$

3.175 n any integer:

$$\sin nx = \sin x\left\{2^{n-1}\cos^{n-1}x - \frac{n-2}{1!}2^{n-3}\cos^{n-3}x + \frac{(n-3)(n-4)}{2!}2^{n-5}\cos^{n-5}x - \frac{(n-4)(n-5)(n-6)}{3!}2^{n-7}\cos^{n-7}x + \ldots\right\}.$$

$$\cos nx = 2^{n-1}\cos^n x - \frac{n}{1!}2^{n-3}\cos^{n-2}x + \frac{n(n-3)}{2!}2^{n-5}\cos^{n-4}x - \frac{n(n-4)(n-5)}{3!}2^{n-7}\cos^{n-6}x + \ldots$$

3.176
$$\sin 2x = 2 \sin x \cos x.$$
$$\sin 3x = \sin x(3 - 4 \sin^2 x)$$
$$= \sin x(4 \cos^2 x - 1).$$
$$\sin 4x = \sin x(8 \cos^3 x - 4 \cos x).$$
$$\sin 5x = \sin x(5 - 20 \sin^2 x + 16 \sin^4 x)$$
$$= \sin x(16 \cos^4 x - 12 \cos^2 x + 1).$$
$$\sin 6x = \sin x(32 \cos^5 x - 32 \cos^3 x + 6 \cos x).$$

3.177
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= 1 - 2 \sin^2 x$$
$$= 2 \cos^2 x - 1.$$
$$\cos 3x = \cos x(4 \cos^2 x - 3)$$
$$= \cos x(1 - 4 \sin^2 x).$$
$$\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.$$
$$\cos 5x = \cos x(16 \cos^4 x - 20 \cos^2 x + 5)$$
$$= \cos x(16 \sin^4 x - 12 \sin^2 x + 1).$$
$$\cos 6x = 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1.$$

3.178
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$
$$\cot 2x = \frac{\cot^2 x - 1}{2 \cot x}.$$

3.180 Integral Powers of Sine and Cosine.

3.181 n an even integer:

$$\sin^n x = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} \left\{ \cos nx - n \cos (n-2)x + \frac{n(n-1)}{2!} \cos (n-4)x \right.$$
$$\left. - \frac{n(n-1)(n-2)}{3!} \cos (n-6)x + \ldots\ldots\ldots + (-1)^{\frac{n}{2}} \frac{1}{2} \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!} \right\}$$

$$\cos^n x = \frac{1}{2^{n-1}} \left\{ \cos nx + n \cos (n-2)x + \frac{n(n-1)}{2!} \cos (n-4)x \right.$$
$$\left. + \frac{n(n-1)(n-2)}{3!} \cos (n-6)x + \ldots + \frac{1}{2} \frac{n!}{\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}. \right\}$$

3.182 n an odd integer:

$$\sin^n x = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \left\{ \sin nx - n \sin (n-2)x + \frac{n(n-1)}{2!} \sin (n-4)x \right.$$

$$\left. - \frac{n(n-1)(n-2)}{3!} \sin (n-6)x + \dots + (-1)^{\frac{n-1}{2}} \frac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} \sin x \right\}.$$

$$\cos^n x = \frac{1}{2^{n-1}} \left\{ \cos nx + n \cos (n-2)x + \frac{n(n-1)}{2!} \cos (n-4)x \right.$$

$$\left. + \frac{n(n-1)(n-2)}{3!} \cos (n-6)x + \dots + \frac{n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!} \cos x \right\}.$$

3.183

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

$$\sin^3 x = \tfrac{1}{4}(3 \sin x - \sin 3x).$$

$$\sin^4 x = \tfrac{1}{8}(\cos 4x - 4 \cos 2x + 3).$$

$$\sin^5 x = \tfrac{1}{16}(\sin 5x - 5 \sin 3x + 10 \sin x).$$

$$\sin^6 x = -\tfrac{1}{32}(\cos 6x - 6 \cos 4x + 15 \cos 2x - 10).$$

3.184

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x).$$

$$\cos^3 x = \tfrac{1}{4}(3 \cos x + \cos 3x).$$

$$\cos^4 x = \tfrac{1}{8}(3 + 4 \cos 2x + \cos 4x).$$

$$\cos^5 x = \tfrac{1}{16}(10 \cos x + 5 \cos 3x + \cos 5x).$$

$$\cos^6 x = \tfrac{1}{32}(10 + 15 \cos 2x + 6 \cos 4x + \cos 6x).$$

INVERSE CIRCULAR FUNCTIONS

3.20 The inverse circular and logarithmic functions are multiple valued; i.e., if

$$0 \leqq \sin^{-1} x \leqq \frac{\pi}{2},$$

the solution of $x = \sin \theta$ is:

$$\theta = 2n\pi + \sin^{-1} x,$$

where n is a positive integer. In the following formulas the cyclic constants are omitted.

3.21

$$\sin^{-1} x = -\sin^{-1}(-x) = \frac{\pi}{2} - \cos^{-1} x = \cos^{-1}\sqrt{1-x^2}$$

$$= \frac{\pi}{2} - \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{4} + \frac{1}{2}\sin^{-1}(2x^2-1)$$

$$= \frac{1}{2}\cos^{-1}(1-2x^2) = \tan^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= 2\tan^{-1}\left\{\frac{1-\sqrt{1-x^2}}{x}\right\} = \frac{1}{2}\tan^{-1}\left\{\frac{2x\sqrt{1-x^2}}{1-2x^2}\right\}$$

$$= \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \frac{\pi}{2} - i\log\left(x+\sqrt{x^2-1}\right).$$

3.22

$$\cos^{-1} x = \pi - \cos^{-1}(-x) = \frac{\pi}{2} - \sin^{-1} x = \frac{1}{2}\cos^{-1}(2x^2-1)$$

$$= 2\cos^{-1}\sqrt{\frac{1+x}{2}} = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= 2\tan^{-1}\sqrt{\frac{1-x}{1+x}} = \frac{1}{2}\tan^{-1}\left\{\frac{2x\sqrt{1-x^2}}{2x^2-1}\right\} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= i\log\left(x+\sqrt{x^2-1}\right) = \pi - i\log\left(\sqrt{x^2-1}-x\right).$$

3.23

$$\tan^{-1} x = -\tan^{-1}(-x) = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{1+x^2} = \frac{\pi}{2} - \cot^{-1} x = \sec^{-1}\sqrt{1+x^2}$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{1}{x} = \frac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}$$

$$= 2\cos^{-1}\left\{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}} = 2\sin^{-1}\left\{\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{2}\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}$$

$$= -\tan^{-1} c + \tan^{-1}\frac{x+c}{1-cx}$$

3.25

1. $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\}$.

2. $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)(1-y^2)}\}$.

3. $\sin^{-1} x \pm \cos^{-1} y = \sin^{-1}\{xy \pm \sqrt{(1-x^2)(1-y^2)}\}$
$$= \cos^{-1}\{y\sqrt{1-x^2} \mp x\sqrt{1-y^2}\}.$$

4. $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\dfrac{x \pm y}{1 \mp xy}$.

5. $\tan^{-1} x \pm \cot^{-1} y = \tan^{-1}\dfrac{xy \pm 1}{y \mp x}$
$$= \cot^{-1}\frac{y \mp x}{xy \pm 1}.$$

HYPERBOLIC FUNCTIONS

3.30 Formulas for the hyperbolic functions may be obtained from the corresponding formulas for the circular functions by replacing x by ix and using the following relations:

1. $\sin ix = \frac{1}{2}i(e^x - e^{-x}) = i \sinh x$.

2. $\cos ix = \frac{1}{2}(e^x + e^{-x}) = \cosh x$.

3. $\tan ix = \dfrac{i(e^{2x} - 1)}{e^{2x} + 1} = i \tanh x$.

4. $\cot ix = -i\dfrac{e^{2x} + 1}{e^{2x} - 1} = -i \coth x$.

5. $\sec ix = \dfrac{2}{e^x + e^{-x}} = \operatorname{sech} x$.

6. $\csc ix = -\dfrac{2i}{e^x - e^{-x}} = -i \operatorname{csch} x$.

7. $\sin^{-1} ix = i \sinh^{-1} x = i \log (x + \sqrt{1 + x^2})$.

8. $\cos^{-1} ix = -i \cosh^{-1} x = \dfrac{\pi}{2} - i \log (x + \sqrt{1 + x^2})$.

9. $\tan^{-1} ix = i \tanh^{-1} x = i \log \sqrt{\dfrac{1 + x}{1 - x}}$.

10. $\cot^{-1} ix = -i \coth^{-1} x = -i \log \sqrt{\dfrac{x + 1}{x - 1}}$.

3.310 The values of five hyperbolic functions in terms of the sixth are given in the following table:

	$\sinh x = a$	$\cosh x = a$	$\tanh x = a$	$\coth x = a$	$\operatorname{sech} x = a$	$\operatorname{csch} x = a$
$\sinh x =$	a	$\sqrt{a^2-1}$	$\dfrac{a}{\sqrt{1-a^2}}$	$\dfrac{1}{\sqrt{a^2-1}}$	$\dfrac{\sqrt{1-a^2}}{a}$	$\dfrac{1}{a}$
$\cosh x =$	$\sqrt{1+a^2}$	a	$\dfrac{1}{\sqrt{1-a^2}}$	$\dfrac{a}{\sqrt{a^2-1}}$	$\dfrac{1}{a}$	$\dfrac{\sqrt{1+a^2}}{a}$
$\tanh x =$	$\dfrac{a}{\sqrt{1+a^2}}$	$\dfrac{\sqrt{a^2-1}}{a}$	a	$\dfrac{1}{a}$	$\sqrt{1-a^2}$	$\dfrac{1}{\sqrt{1+a^2}}$
$\coth x =$	$\dfrac{\sqrt{a^2+1}}{a}$	$\dfrac{a}{\sqrt{a^2-1}}$	$\dfrac{1}{a}$	a	$\dfrac{1}{\sqrt{1-a^2}}$	$\sqrt{1+a^2}$
$\operatorname{sech} x =$	$\dfrac{1}{\sqrt{1+a^2}}$	$\dfrac{1}{a}$	$\sqrt{1-a^2}$	$\dfrac{\sqrt{a^2-1}}{a}$	a	$\dfrac{a}{\sqrt{1+a^2}}$
$\operatorname{csch} x =$	$\dfrac{1}{a}$	$\dfrac{1}{\sqrt{a^2-1}}$	$\dfrac{\sqrt{1-a^2}}{a}$	$\sqrt{a^2-1}$	$\dfrac{a}{\sqrt{1-a^2}}$	a

3.311 Periodicity of the Hyperbolic Functions.

The functions $\sinh x$, $\cosh x$, $\operatorname{sech} x$, $\operatorname{csch} x$ have an imaginary period $2\pi i$, e.g.:

$$\cosh x = \cosh (x + 2\pi i n),$$

where n is any integer. The functions $\tanh x$, $\coth x$ have an imaginary period πi.

The values of the hyperbolic functions for the argument 0, $\frac{\pi}{2}i$, πi, $\frac{3\pi i}{2}$, are given in the following table:

	0	$\frac{\pi}{2}i$	πi	$\frac{3\pi}{2}i$
sinh	0	i	0	$-i$
cosh	1	0	-1	0
tanh	0	$\infty \cdot i$	0	$\infty \cdot i$
coth	∞	0	∞	0
sech	1	∞	-1	∞
csch	∞	$-i$	∞	i

3.320

1. $$\sinh \tfrac{1}{2}x = \sqrt{\frac{\cosh x - 1}{2}}.$$

2. $$\cosh \tfrac{1}{2}x = \sqrt{\frac{\cosh x + 1}{2}}.$$

3. $$\tanh \tfrac{1}{2}x = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1} = \sqrt{\frac{\cosh x - 1}{\cosh x + 1}}.$$

3.33

1. $$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$$

2. $$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$$

3. $$\tanh (x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}.$$

4. $$\coth (x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x}.$$

3.34

1. $$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y).$$

2. $$\sinh x - \sinh y = 2 \cosh \tfrac{1}{2}(x + y) \sinh \tfrac{1}{2}(x - y).$$

3. $$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y).$$

4. $$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2}(x + y) \sinh \tfrac{1}{2}(x - y).$$

5. $$\tanh x + \tanh y = \frac{\sinh (x + y)}{\cosh x \cosh y}.$$

6. $$\tanh x - \tanh y = \frac{\sinh (x - y)}{\cosh x \cosh y}.$$

7. $$\coth x + \coth y = \frac{\sinh (x + y)}{\sinh x \sinh y}.$$

8. $$\coth x - \coth y = -\frac{\sinh (x - y)}{\sinh x \sinh y}.$$

3.35

1. $$\sinh(x+y) + \sinh(x-y) = 2\sinh x\cosh y.$$
2. $$\sinh(x+y) - \sinh(x-y) = 2\cosh x\sinh y.$$
3. $$\cosh(x+y) + \cosh(x-y) = 2\cosh x\cosh y.$$
4. $$\cosh(x+y) - \cosh(x-y) = 2\sinh x\sinh y.$$
5. $$\tanh\tfrac{1}{2}(x \pm y) = \frac{\sinh x \pm \sinh y}{\cosh x + \cosh y}.$$
6. $$\coth\tfrac{1}{2}(x \pm y) = \frac{\sinh x \pm \sinh y}{\cosh x - \cosh y}.$$
7. $$\frac{\tanh x + \tanh y}{\tanh x - \tanh y} = \frac{\sinh(x+y)}{\sinh(x-y)}.$$
8. $$\frac{\coth x + \coth y}{\coth x - \coth y} = -\frac{\sinh(x+y)}{\sinh(x-y)}.$$

3.36

1. $$\sinh(x+y) + \cosh(x+y) = (\cosh x + \sinh x)(\cosh y + \sinh y).$$
2. $$\sinh(x+y)\sinh(x-y) = \sinh^2 x - \sinh^2 y$$
$$= \cosh^2 x - \cosh^2 y.$$
3. $$\cosh(x+y)\cosh(x-y) = \cosh^2 x + \sinh^2 y$$
$$= \sinh^2 x + \cosh^2 y.$$
4. $$\sinh x + \cosh x = \frac{1 + \tanh\frac{1}{2}x}{1 - \tanh\frac{1}{2}x}.$$
5. $$(\sinh x + \cosh x)^n = \cosh nx + \sinh nx.$$

3.37

1. $$e^x = \cosh x + \sinh x.$$
2. $$e^{-x} = \cosh x - \sinh x.$$
3. $$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$
4. $$\cosh x = \tfrac{1}{2}(e^x + e^{-x}).$$

3.38

1. $$\sinh 2x = 2 \sinh x \cosh x, \quad = \frac{2 \tanh x}{1 - \tanh^2 x}.$$

2. $$\cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1, \quad = 1 + 2 \sinh^2 x, \quad = \frac{1 + \tanh^2 x}{1 - \tanh^2 x}.$$

3. $$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

4. $$\sinh 3x = 3 \sinh x + 4 \sinh^3 x.$$

5. $$\cosh 3x = 4 \cosh^3 x - 3 \cosh x.$$

6. $$\tanh 3x = \frac{3 \tanh x + \tanh^3 x}{1 + 3 \tanh^2 x}.$$

3.40 Inverse Hyperbolic Functions.

The hyperbolic functions being periodic, the inverse functions are multiple valued (**3.311**). In the following formulas the periodic constants are omitted, the principal values only being given.

1. $$\sinh^{-1} x = \log (x + \sqrt{x^2 + 1}) = \cosh^{-1} \sqrt{x^2 + 1}.$$

2. $$\cosh^{-1} x = \log (x + \sqrt{x^2 - 1}) = \sinh^{-1} \sqrt{x^2 - 1}.$$

3. $$\tanh^{-1} x = \log \sqrt{\frac{1 + x}{1 - x}}.$$

4. $$\coth^{-1} x = \log \sqrt{\frac{x + 1}{x - 1}} = \tanh^{-1} \frac{1}{x}.$$

5. $$\operatorname{sech}^{-1} x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1} \right) = \cosh^{-1} \frac{1}{x}.$$

6. $$\operatorname{csch}^{-1} x = \log \left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1} \right) = \sinh^{-1} \frac{1}{x}.$$

3.41

1. $$\sinh^{-1} x \pm \sinh^{-1} y = \sinh^{-1} (x\sqrt{1 + y^2} \pm y\sqrt{1 + x^2}).$$

2. $$\cosh^{-1} x \pm \cosh^{-1} y = \cosh^{-1} (xy \pm \sqrt{(x^2 - 1)(y^2 - 1)}).$$

3. $$\tanh^{-1} x \pm \tanh^{-1} y = \tanh^{-1} \frac{x \pm y}{1 \pm xy}.$$

3.42

1. $$\cosh^{-1}\frac{1}{2}\left(x+\frac{1}{x}\right)=\sinh^{-1}\frac{1}{2}\left(x-\frac{1}{x}\right),$$
$$=\tanh^{-1}\frac{x^2-1}{x^2+1}=2\tanh^{-1}\frac{x-1}{x+1},$$
$$=\log x.$$

2. $$\cosh^{-1}\csc 2x=-\sinh^{-1}\cot 2x=-\tanh^{-1}\cos 2x,$$
$$=-\log\tan x.$$

3. $$\tanh^{-1}\tan^2\left(\frac{\pi}{4}+\frac{x}{2}\right)=\frac{1}{2}\log\csc x.$$

4. $$\tanh^{-1}\tan^2\frac{x}{2}=\frac{1}{2}\log\sec x.$$

3.43 The Gudermannian.

If,

1. $$\cosh x=\sec\theta.$$

2. $$\sinh x=\tan\theta.$$

3. $$e^x=\sec\theta+\tan\theta=\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

4. $$x=\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

5. $$\theta=\operatorname{gd} x.$$

3.44

1. $$\sinh x=\tan\operatorname{gd} x.$$

2. $$\cosh x=\sec\operatorname{gd} x.$$

3. $$\tanh x=\sin\operatorname{gd} x.$$

4. $$\tanh\frac{x}{2}=\tan\frac{1}{2}\operatorname{gd} x.$$

5. $$e^x=\frac{1+\sin\operatorname{gd} x}{\cos\operatorname{gd} x}=\frac{1-\cos\left(\frac{\pi}{2}+\operatorname{gd} x\right)}{\sin\left(\frac{\pi}{2}+\operatorname{gd} x\right)}.$$

	γ	$\gamma = 180^\circ - (\alpha + \beta).$
	c	$c = \dfrac{a \sin \gamma}{\sin \alpha} = \dfrac{a \sin (\alpha + \beta)}{\sin \alpha}.$

7. $\tan^{-1} \tanh x = \frac{1}{2} \operatorname{gd}^{-1} 2x.$

3.50 SOLUTION OF OBLIQUE PLANE TRIANGLES

$a, b, c =$ Sides of triangle,

$\alpha, \beta, \gamma =$ angles opposite to a, b, c, respectively,

$A =$ area of triangle,

$s = \frac{1}{2}(a + b + c).$

Given	*Sought*	*Formula*
a, b, c	α	$\sin \frac{1}{2}\alpha = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$
		$\cos \frac{1}{2}\alpha = \sqrt{\dfrac{s(s-a)}{bc}}.$
		$\tan \frac{1}{2}\alpha = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}.$
		$\cos \alpha = \dfrac{c^2 + b^2 - a^2}{2bc}.$
	A	$A = \sqrt{s(s-a)(s-b)(s-c)}.$
a, b, α	β	$\sin \beta = \dfrac{b \sin \alpha}{a}.$
		When $a > b$, $\beta < \frac{\pi}{2}$ and but one value results. When $b >$
		β has two values.
	γ	$\gamma = 180^\circ - (\alpha + \beta).$
	c	$c = \dfrac{a \sin \gamma}{\sin \alpha}.$
	A	$A = \frac{1}{2} ab \sin \gamma.$
a, α, β	b	$b = \dfrac{a \sin \beta}{\sin \alpha}$
	γ	$\gamma = 180^\circ - (\alpha + \beta).$
	c	$c = \dfrac{a \sin \gamma}{\sin \alpha} = \dfrac{a \sin (\alpha + \beta)}{\sin \alpha}.$

Given	*Sought*	*Formula*
	A	$A = \frac{1}{2} ab \sin\gamma = \frac{1}{2} a^2 \frac{\sin\beta \sin\gamma}{\sin\alpha}$.
a, b, γ	α	$\tan\alpha = \frac{a \sin\gamma}{b - a\cos\gamma}$.
	α, β	$\frac{1}{2}(\alpha+\beta) = 90^\circ - \frac{1}{2}\gamma$.
		$\tan\frac{1}{2}(\alpha-\beta) = \frac{a-b}{a+b}\cot\frac{1}{2}\gamma$
	c	$c = (a^2+b^2-2ab\cos\gamma)^{\frac{1}{2}}$.
		$= \{(a+b)^2 - 4ab\cos^2\frac{1}{2}\gamma\}^{\frac{1}{2}}$
		$= \{(a-b)^2 + 4ab\sin^2\frac{1}{2}\gamma\}^{\frac{1}{2}}$.
		$= \frac{a-b}{\cos\phi}$ where $\tan\phi = 2\sqrt{ab}\,\frac{\sin\frac{1}{2}\gamma}{a-b}$
		$= \frac{a\sin\gamma}{\sin\alpha}$.
	A	$A = \frac{1}{2} ab \sin\gamma$.

SOLUTION OF SPHERICAL TRIANGLES

3.51 Right-angled spherical triangles.

a, b, c = sides of triangle, c the side opposite γ, the right angle.

α, β, γ = angles opposite a, b, c, respectively.

3.511 Napier's Rules:

The five parts are a, b, $co\ c$, $co\ \alpha$, $co\ \beta$, where $co\ c = \frac{\pi}{2} - c$. The right ang γ is omitted.

The sine of the middle part is equal to the product of the tangents of t adjacent parts.

The sine of the middle part is equal to the product of the cosines of opposi parts.

From these rules the following equations follow:

$$\sin a = \sin c \sin\alpha,$$
$$\tan a = \tan c \cos\beta = \sin b \tan\alpha,$$
$$\sin b = \sin c \sin\beta,$$
$$\tan b = \tan c \cos\alpha = \sin a \tan\beta,$$
$$\cos\alpha = \cos a \sin\beta,$$
$$\cos\beta = \cos b \sin\alpha,$$
$$\cos c = \cot\alpha \cot\beta = \cos a \cos b.$$

3.62 Oblique-angled spherical triangles.

a, b, c = sides of triangle.

α, β, γ = angles opposite to a, b, c, respectively.

$s = \frac{1}{2}(a + b + c)$,

$\sigma = \frac{1}{2}(\alpha + \beta + \gamma)$,

$\epsilon = \alpha + \beta + \gamma - 180$ = spherical excess,

S = surface of triangle on sphere of radius r.

Given	*Sought*	*Formula*
a, b, c	α	$\sin^2 \frac{1}{2}\alpha = \text{haversin } \alpha$, $= \dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$
		$\tan^2 \frac{1}{2}\alpha = \dfrac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}$.
		$\cos^2 \frac{1}{2}\alpha = \dfrac{\sin s \sin(s-a)}{\sin b \sin c}$.
		$\text{haversin } \alpha = \dfrac{\text{hav } a - \text{hav}(b-c)}{\sin b \sin c}$.
α, β, γ	a	$\sin^2 \frac{1}{2}a = \text{haversin } a$, $= \dfrac{-\cos\sigma\cos(\sigma-\alpha)}{\sin\beta\sin\gamma}$
		$\tan^2 \frac{1}{2}a = \dfrac{-\cos\sigma\cos(\sigma-\alpha)}{\cos(\sigma-\beta)\cos(\sigma-\gamma)}$.
		$\cos^2 \frac{1}{2}a = \dfrac{\cos(\sigma-\beta)\cos(\sigma-\gamma)}{\sin\beta\sin\gamma}$.
a, c, α Ambiguous case. Two solutions possible.	γ	$\sin\gamma = \dfrac{\sin\alpha\sin c}{\sin a}$.
	β	$\left\{\begin{array}{l}\tan\theta = \tan\alpha\cos c.\\ \sin(\beta+\theta) = \sin\theta\tan c\cot a\end{array}\right.$
	b	$\left\{\begin{array}{l}\cot\phi = \tan c\cos\alpha.\\ \sin(b+\phi) = \dfrac{\cos a\sin\phi}{\cos c}.\end{array}\right.$
α, γ, c Ambiguous case. Two solutions possible.	c	$\sin c = \dfrac{\sin a\sin\gamma}{\sin\alpha}$.

Given	*Sought*	*Formula*
	b	$\tan \theta = \tan a \cos \gamma.$ $\sin (b - \theta) = \cot \alpha \tan \gamma \sin \theta.$
	b	$\tan \frac{1}{2} b = \dfrac{\sin \frac{1}{2}(\alpha + \gamma)}{\sin \frac{1}{2}(\alpha - \gamma)} \tan \frac{1}{2}(a - c)$ $= \dfrac{\cos \frac{1}{2}(\alpha + \gamma)}{\cos \frac{1}{2}(\alpha - \gamma)} \tan \frac{1}{2}(a + c).$
	β	$\cot \phi = \cos a \tan \gamma$ $\sin (\beta - \phi) = \dfrac{\cos \alpha \sin \phi}{\cos \gamma}.$
	β	$\cot \frac{1}{2} \beta = \dfrac{\sin \frac{1}{2}(a + c)}{\sin \frac{1}{2}(a - c)} \tan \frac{1}{2}(\alpha - \gamma).$ $= \dfrac{\cos \frac{1}{2}(a + c)}{\cos \frac{1}{2}(a - c)} \tan \frac{1}{2}(\alpha + \gamma).$
a, b, γ	c	$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$
$\tan \theta = \tan a \cos \gamma$ $\tan \phi = \tan b \cos \gamma$	c	$\cos c = \dfrac{\cos a \cos (b - \theta)}{\cos \theta}$ $= \dfrac{\cos b \cos (a - \phi)}{\cos \phi}.$
		$\operatorname{hav} c = \operatorname{hav} (a - b) + \sin a \sin b \operatorname{hav} \gamma$
	α	$\tan \alpha = \dfrac{\sin \theta \tan \gamma}{\sin (b - \theta)}.$
	β	$\sin \beta = \dfrac{\sin \gamma \sin b}{\sin c}.$ $= \dfrac{\sin \alpha \sin b}{\sin a}.$ $\tan \beta = \dfrac{\sin \phi \tan \gamma}{\sin (a - \phi)}.$
	α, β	$\tan \frac{1}{2}(\alpha + \beta) = \dfrac{\cos \frac{1}{2}(a - b) \cot \frac{1}{2}\gamma}{\cos \frac{1}{2}(a + b)}$ $\tan \frac{1}{2}(\alpha - \beta) = \dfrac{\sin \frac{1}{2}(a - b) \cot \frac{1}{2}\gamma}{\sin \frac{1}{2}(a + b)}.$
c, α, β	γ	$\cos \gamma = - \cos \alpha \cos \beta + \sin \alpha \sin \beta \; \mathrm{co}$
$\tan \theta = \cos c \tan \alpha$ $\tan \phi = \cos c \tan \beta$		$\cos \gamma = \dfrac{\cos \alpha \cos (\beta + \theta)}{\cos \theta}.$ $= \dfrac{\cos \beta \cos (\alpha + \phi)}{\cos \phi}.$
	a	$\tan a = \dfrac{\tan c \sin \theta}{\sin (\beta + \theta)}.$

Given	*Sought*	*Formula*
	b	$\tan b = \dfrac{\tan c \sin \phi}{\sin(\alpha + \phi)}$.
	a, b	$\begin{cases} \tan \frac{1}{2}(a + b) = \dfrac{\cos \frac{1}{2}(\alpha - \beta) \tan \frac{1}{2}c}{\cos \frac{1}{2}(\alpha + \beta)} \\ \tan \frac{1}{2}(a - b) = \dfrac{\sin \frac{1}{2}(\alpha - \beta) \tan \frac{1}{2}c}{\sin \frac{1}{2}(\alpha + \beta)}. \end{cases}$
a, b, γ	e	$\cot \frac{1}{2}e = \dfrac{\cot \frac{1}{2}a \cot \frac{1}{2}b + \cos \gamma}{\sin \gamma}$.
a, b, c	e	$\tan^2 \frac{1}{4}e = \tan \frac{1}{2}s \tan \frac{1}{2}(s - a) \tan \frac{1}{2}(s - b) \tan \frac{1}{2}(s - c)$.
e, γ	S	$S = \dfrac{e}{180^\circ} \pi r^2$.

FINITE SERIES OF CIRCULAR FUNCTIONS

3.60 If the sum, $f(r)$, of the finite or infinite series:

$$f(r) = a_0 + a_1 r + a_2 r^2 + \dots\dots$$

is known, the sums of the series:

$$S_1 = a_0 \cos x + a_1 r \cos(x + y) + a_2 r^2 \cos(x + 2y) + \dots\dots$$

$$S_2 = a_0 \sin x + a_1 r \sin(x + y) + a_2 r^2 \sin(x + 2y) + \dots\dots$$

are:

$$S_1 = \tfrac{1}{2}\{e^{ix} f(re^{iy}) + e^{-ix} f(re^{-iy})\},$$

$$S_2 = -\frac{i}{2}\{e^{ix} f(re^{iy}) - e^{-ix} f(re^{-iy})\}.$$

3.61 Special Finite Series.

1. $$\sum_{k=1}^{n} \sin kx = \frac{\sin \frac{nx}{2} \sin \frac{n+1}{2} x}{\sin \frac{x}{2}}.$$

2. $$\sum_{k=0}^{n} \cos kx = \frac{\cos \frac{nx}{2} \sin \frac{n+1}{2} x}{\sin \frac{x}{2}}.$$

3. $$\sum_{k=1}^{n} \sin^2 kx = \frac{n}{2} - \frac{\cos (n+1)x \sin nx}{2 \sin x}.$$

4. $$\sum_{k=0}^{n} \cos^2 kx = \frac{n+2}{2} + \frac{\cos (n+1)x \sin nx}{2 \sin x}.$$

5. $$\sum_{k=1}^{n-1} k \sin kx = \frac{\sin nx}{4 \sin^2 \frac{x}{2}} - \frac{n \cos \left(\frac{2n-1}{2}\right)x}{2 \sin \frac{x}{2}}.$$

6. $$\sum_{k=1}^{n-1} k \cos kx = \frac{n \sin \left(\frac{2n-1}{2}\right)x}{2 \sin \frac{x}{2}} - \frac{1 - \cos nx}{4 \sin^2 \frac{x}{2}}.$$

7. $$\sum_{k=1}^{n} \sin (2k-1)x = \frac{\sin^2 nx}{\sin x}.$$

8. $$\sum_{k=0}^{n} \sin (x+ky) = \frac{\sin \left(x + \frac{ny}{2}\right) \sin \left(\frac{n+1}{2} y\right)}{\sin \frac{y}{2}}.$$

9. $$\sum_{k=0}^{n} \cos (x+ky) = \frac{\cos \left(x + \frac{n}{2} y\right) \sin \left(\frac{n+1}{2} y\right)}{\sin \frac{y}{2}}.$$

10. $$\sum_{k=1}^{n+1} (-1)^{k-1} \sin (2k-1)x = (-1)^n \frac{\sin (2n+2)x}{2 \cos x}.$$

11. $$\sum_{k=1}^{n} (-1)^k \cos kx = -\frac{1}{2} + (-1)^n \frac{\cos \left(\frac{2n+1}{2} x\right)}{2 \cos \frac{x}{2}}.$$

12. $$\sum_{k=1}^{n-1} r^k \sin kx = \frac{r \sin x(1 - r^n \cos nx) - (1 - r \cos x) r^n \sin nx}{1 - 2r \cos x + r^2}.$$

13. $$\sum_{k=0}^{n-1} r^k \cos kx = \frac{(1 - r \cos x)(1 - r^n \cos nx) + r^{n+1} \sin x \sin nx}{1 - 2r \cos x + r^2}.$$

14. $$\sum_{k=1}^{n} \left(\frac{1}{2^k} \sec \frac{x}{2^k}\right)^2 = \csc^2 x - \left(\frac{1}{2^n} \csc \frac{x}{2^n}\right)^2.$$

15. $$\sum_{k=1}^{n} \left(2^k \sin^2 \frac{x}{2^k}\right)^2 = \left(2^n \sin \frac{x}{2^n}\right)^2 - \sin^2 x.$$

16. $$\sum_{k=0}^{n} \frac{1}{2^k}\tan\frac{x}{2^k} = \frac{1}{2^n}\cot\frac{x}{2^n} - 2\cot 2x.$$

17. $$\sum_{k=0}^{n-1} \cos\frac{k^2 2\pi}{n} = \frac{\sqrt{n}}{2}\left(1+\cos\frac{n\pi}{2}+\sin\frac{n\pi}{2}\right).$$

18. $$\sum_{k=1}^{n-1} \sin\frac{k^2 2\pi}{n} = \frac{\sqrt{n}}{2}\left(1+\cos\frac{n\pi}{2}-\sin\frac{n\pi}{2}\right).$$

19. $$\sum_{k=1}^{n-1} \sin\frac{k\pi}{n} = \cot\frac{\pi}{2n}.$$

20. $$\sum_{k=0}^{n} \frac{1}{2^{2k}}\tan^2\frac{x}{2^k} = \frac{2^{2n+2}-1}{3\cdot 2^{2n-1}} + 4\cot^2 2x - \frac{1}{2^{2n}}\cot\frac{x}{2^n}.$$

3.02

$$S_n = \sum_{k=1}^{n-1} \csc\frac{k\pi}{n}.$$

Watson (Phil. Mag. 31, p. 111, 1916) has obtained an asymptotic expan
for this sum, and has given the following approximation:

$S_n = 2n[0.7329355992 \log_{10}(2n) - 0.1806453871]$

$$-\frac{0.087266}{n} + \frac{0.01035}{n^3} - \frac{0.004}{n^5} + \frac{0.005}{n^7} - \cdots.$$

Values of S_n are tabulated by integers from $n = 2$ to $n = 30$, and from $n =$
to $n = 100$ at intervals of 5.

The expansion of

$$T_n = \sum_{k=1}^{n-1} \csc\left(\frac{k\pi}{n} - \frac{\beta}{2}\right),$$

where

$$-\frac{2\pi}{n} < \beta < \frac{2\pi}{n},$$

is also obtained.

3.70 Finite Products.

1. $$\sin nx = n \sin x \cos x \prod_{k=1}^{\frac{n}{2}-1} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \quad n \text{ even.}$$

2. $$\cos nx = \prod_{k=1}^{\frac{n}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{2k-1}{2n}\pi}\right) \quad n \text{ even.}$$

3. $$\sin nx = n \sin x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{k\pi}{n}}\right) \quad n \text{ odd.}$$

4. $$\cos nx = \cos x \prod_{k=1}^{\frac{n-1}{2}} \left(1 - \frac{\sin^2 x}{\sin^2 \frac{2k-1}{2n}\pi}\right) \quad n \text{ odd.}$$

5. $$\cos nx - \cos ny = 2^{n-1} \prod_{k=0}^{n-1} \left\{\cos x - \cos\left(y + \frac{2k\pi}{n}\right)\right\}.$$

6. $$a^{2n} - 2a^n b^n \cos nx + b^{2n} = \prod_{k=0}^{n-1} \left\{a^2 - 2ab \cos\left(x + \frac{2k\pi}{n}\right) + b^2\right\}.$$

ROOTS OF TRANSCENDENTAL EQUATIONS

3.800 $\tan x = x$.

The first 17 roots, and the corresponding maxima and minima of $\frac{\sin x}{x}$ are given in the following table (Lommel, Abh. Munch. Akad. (2) 15, 123, 1886):

n	x_n	Max / Min $\frac{\sin x}{x}$
1	0	1
2	4.4934	−0.2172
3	7.7253	+0.1284
4	10.9041	−0.0913
5	14.0662	+0.0709
6	17.2208	−0.0580
7	20.3713	+0.0490
8	23.5195	−0.0425
9	26.6661	+0.0375
10	29.8116	−0.0335
11	32.9564	+0.0303
12	36.1006	−0.0277
13	39.2444	+0.0255
14	42.3879	−0.0236
15	45.5311	+0.0220
16	48.6741	−0.0205
17	51.8170	[illegible]

3.801

$$\tan x = \frac{2x}{2 - x^2}.$$

The first three roots are:

$$x_1 = 0,$$

$$x_2 = 119.26 \frac{\pi}{180},$$

$$x_3 = 340.35 \frac{\pi}{180}.$$

If x is large

$$x_n = n\pi - \frac{2}{n\pi} - \frac{16}{3n^3\pi^3} + \ldots .$$

(Rayleigh, Theory of Sound, II, p. 265.)

3.802

$$\tan x = \frac{x^3 - 9x}{4x^2 - 9}.$$

The first two roots are:

$$x_1 = 0,$$

$$x_2 = 3.3422.$$

(Rayleigh, l. c. p. 266.)

3.803

$$\tan x = \frac{x}{1 - x^2}.$$

The first two roots are:

$$x_1 = 0,$$

$$x_2 = 2.744.$$

(J. J. Thomson, Recent Researches, p. 373.)

3.804

$$\tan x = \frac{3x}{3 - x^2}.$$

The first seven roots are:

$$x_1 = 0,$$

$$x_2 = 1.8346\pi,$$

$$x_3 = 2.8950\pi,$$

$$x_4 = 3.9225\pi,$$

$$x_5 = 4.9385\pi,$$

$$x_6 = 5.9489\pi,$$

$$x_7 = 6.9563\pi.$$

(Lamb, London Math. Soc. Proc. 13, 1882.)

3.805

$$\tan x = \frac{4x}{4 - 3x^2}.$$

The first seven roots are:

$$x_1 = 0,$$
$$x_2 = 0.8160\pi,$$
$$x_3 = 1.0285\pi,$$
$$x_4 = 2.0350\pi,$$
$$x_5 = 3.0658\pi,$$
$$x_6 = 4.0728\pi,$$
$$x_7 = 5.0774\pi.$$

(Lamb, l. c.)

3.806

$$\cos x \cosh x = 1.$$

The roots are:

$$x_1 = 4.7300408,$$
$$x_2 = 7.8532046,$$
$$x_3 = 10.9956078,$$
$$x_4 = 14.1371655,$$
$$x_5 = 17.2787596,$$
$$x_n = \tfrac{1}{2}(2n+1)\pi \quad n > 5.$$

(Rayleigh, Theory of Sound, I, p. 278.)

3.807

$$\cos x \cosh x = -1.$$

The roots are:

$$x_1 = 1.875104,$$
$$x_2 = 4.694098,$$
$$x_3 = 7.854757,$$
$$x_4 = 10.995541,$$
$$x_5 = 14.137168,$$
$$x_6 = 17.278759,$$
$$x_n = \tfrac{1}{2}(2n-1)\pi \quad n > 6.$$

3.808

$$1 - (1 + x^2) \cos x = 0.$$

The roots are:

$$x_1 = 1.102506,$$
$$x_2 = 4.754761,$$
$$x_3 = 7.837964,$$
$$x_4 = 11.003766,$$
$$x_5 = 14.132185,$$
$$x_6 = 17.282097.$$

(Schlömilch: Übungsbuch, I, p. 354.)

3.809 The smallest root of

$$\theta - \cot \theta = 0,$$

is

$$\theta = 49^\circ\, 17'\, 36''.5.$$

3.810 The smallest root of

$$\theta - \cos\theta = 0,$$

is

$$\theta = 42^\circ\, 20'\, 47''.3.$$

(l. c. p. 353.)

3.811 The smallest root of

$$xe^x - 2 = 0,$$

is

$$x = 0.8526.$$

(l. c. p. 353.)

3.812 The smallest root of

$$\log(1 + x) - \tfrac{3}{4}x = 0,$$

is

$$x = 0.73360.$$

(l. c. p. 353.)

3.813

$$\tan x - x + \frac{1}{x} = 0.$$

The first roots are:

$$x_1 = 4.480,$$
$$x_2 = 7.723,$$
$$x_3 = 10.90,$$
$$x_4 = 14.07.$$

(Collo, Annalen der Physik, 65, p. 45, 1921.)

3.814

$$\cot x + x - \frac{1}{x} = 0.$$

The first roots are:

$$x_1 = 0,$$
$$x_2 = 2.744,$$
$$x_3 = 6.117,$$
$$x_4 = 9.317,$$
$$x_5 = 12.48,$$
$$x_6 = 15.64,$$
$$x_7 = 18.80.$$

(Collo, l. c.)

3.90 Special Tables.

$\sin\theta$, $\cos\theta$: The British Association Report for 1916 contains the follow tables:

Table I, p. 60. $\sin\theta$, $\cos\theta$, θ expressed in radians from $\theta = 0$ to $\theta = 1.$ interval 0.001, 10 decimal places.

Table II, p. 88. $\theta - \sin\theta$, $1 - \cos\theta$, $\theta = 0.00001$ to $\theta = 0.00100$, inte

Table III, p. 90. $\sin\theta$, $\cos\theta$; $\theta = 0.1$ to $\theta = 10.0$, interval 0.1, 15 decimal places.

J. Peters (Abh. d. K. P. Akad. der Wissen., Berlin, 1911) has given sines and cosines for every sexagesimal second to 21 places.

hav θ, $\log_{10}$ hav θ: Bowditch, American Practical Navigator, five place tables, 0° – 180°, for 15″ intervals.

Tables for Solution of Spherical Triangles.

Aquino's Altitude and Azimuth Tables, London, 1918. Reprinted in Hydrographic Office Publication, No. 200, Washington, 1918.

Hyperbolic Functions.

The Smithsonian Mathematical Tables: Hyperbolic Functions, contain the most complete five-place tables of Hyperbolic Functions.

Table I. The common logarithms (base 10) of $\sinh u$, $\cosh u$, $\tanh u$, $\coth u$:

$u = 0.0001$ to $u = 0.1000$ interval 0.0001,
$u = 0.001$ to $u = 3.000$ interval 0.001,
$u = 3.00$ to $u = 6.00$ interval 0.01.

Table II. $\sinh u$, $\cosh u$, $\tanh u$, $\coth u$. Same ranges and intervals.

Table III. $\sin u$, $\cos u$, $\log_{10}\sin u$, $\log_{10}\cos u$:

$u = 0.0001$ to $u = 0.1000$ interval 0.0001,
$u = 0.100$ to $u = 1.600$ interval 0.001.

Table IV. $\log_{10}e^u$ (7 places), e^u and e^{-u} (7 significant figures):

$u = 0.001$ to $u = 2.950$ interval 0.001,
$u = 3.00$ to $u = 6.00$ interval 0.01,
$u = 1.0$ to $u = 100$ interval 1.0 (to 10 figures).

Table V. five-place table of natural logarithms, $\log u$.

$u = 1.0$ to $u = 1000$ interval 1.0,
$u = 1000$ to $u = 10{,}000$ varying intervals.

Table VI. $gd\,u$ (7 places); u expressed in radians, $u = 0.001$ to $u = 3.000$, interval 0.001, and the corresponding angular measure. $u = 3.00$ to $u = 6.00$, interval 0.01.

Table VII. $gd^{-1}u$, to 0′.01, in terms of $gd\,u$ in degrees and minutes from 0° 1′ to 89° 50′.

Kennelly: Tables of Complex Hyperbolic and Circular Functions. Cambridge, Harvard University Press, 1914.

The complex argument, $x + iq = \rho e^{i\delta}$. In the tables this is denoted $\rho\angle\delta$. $\rho = \sqrt{x^2 + q^2}$, $\tan \delta = q/x$.

Tables I, II, III give the hyperbolic sine, cosine and tangent of $(\rho\angle\delta)$ expressed as $r\angle\gamma$:

$\delta = 45^\circ$ to $\delta = 90^\circ$ interval 1°
$\rho = 0.01$ to $\rho = 3.0$ interval 0.1.

Tables IV and V give $\frac{\sinh\theta}{\theta}$, $\frac{\tanh\theta}{\theta}$ expressed as $r\angle\gamma$, $\theta = \rho\angle\delta$,

$\rho = 0.1$ to $\rho = 3.0$ interval 0.1,
$\delta = 45^\circ$ to $\delta = 90^\circ$ interval 1°.

Table VI gives sinh $(\rho\angle 45^\circ)$, cosh $(\rho\angle 45^\circ)$, tanh $(\rho\angle 45^\circ)$, coth $(\rho\angle 45^\circ)$, sech $(\rho\angle 45^\circ)$, csch $(\rho\angle 45^\circ)$ expressed as $r\angle\gamma$:

$\rho = 0$ to $\rho = 6.0$ interval 0.1,
$\rho = 6.05$ to $\rho = 20.50$ interval 0.05.

Tables VII, VIII and IX give sinh $(x + iq)$, cosh $(x + iq)$, tanh $(x + iq)$, expressed as $u + iv$:

$x = 0$ to $x = 3.05$ interval 0.05,
$q = 0$ to $q = 2.0$ interval 0.05.

Tables X, XI, XII give sinh $(x + iq)$, cosh $(x + iq)$, tanh $(x + iq)$ expressed as $r\angle\gamma$:

$x = 0$ to $x = 3.05$ interval 0.05,
$q = 0$ to $q = 2.0$ interval 0.05.

Table XIII gives sinh $(4 + iq)$, cosh $(4 + iq)$, tanh $(4 + iq)$ expressed both as $u + iv$ and $r\angle\gamma$:

$q = 0$ to $q = 2.0$ interval 0.05.

Table XIV gives $\frac{e^x}{2}$ and $\log_{10}\frac{e^x}{2}$.

$x = 4.00$ to $x = 10.00$ interval 0.01.

Table XV gives the real hyperbolic functions: sinh θ, cosh θ, tanh θ, coth θ, sech θ, csch θ.

$\theta = 0$ to $\theta = 2.5$ interval 0.01,
$\theta = 2.5$ to $\theta = 7.5$ interval 0.1.

Pernot and Woods: Logarithms of Hyperbolic Functions to 12 Significant Figures. Berkeley, University of California Press, 1918.

Table I. $\log_{10} \sinh x$, with the first three differences.

$x = .0000$ to $x = 2.018$ interval 0.0001.

Table II. $\log_{10} \cosh x$.

$x = 0.000$ to $x = 2.032$ interval 0.001.

Table III. $\log_{10} \tanh x$.

$x = 0.000$ to $x = 2.018$ interval 0.001.

Table IV. $\log_{10} \dfrac{\sinh x}{x}$.

$x = 0.00$ to $x = 0.506$ interval 0.001.

Table V. $\log_{10} \dfrac{\tanh x}{x}$.

$x = 0.000$ to $x = 0.506$ interval 0.001.

Van Orstrand, Memoirs of the National Academy of Sciences, Vol. XIV, fifth memoir, Washington, 1921.

Tables of $\frac{1}{n!}$, e^x, e^{-x}, $e^{n\pi}$, $e^{-n\pi}$, $e^{\pm\frac{n\pi}{4}}$, $\sin x$, $\cos x$, to 23 to 33 decimal places or significant figures.

IV. VECTOR ANALYSIS

4.000 A vector **A** has components along the three rectangular axes, x, y, z: A_x, A_y, A_z.

$$A = \text{length of vector.}$$
$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}.$$

Direction cosines of **A**, $\dfrac{A_x}{A}, \dfrac{A_y}{A}, \dfrac{A_z}{A}$.

4.001 Addition of vectors.

$$\mathbf{A} + \mathbf{B} = \mathbf{C}.$$

C is a vector with components,

$$C_x = A_x + B_x.$$
$$C_y = A_y + B_y.$$
$$C_z = A_z + B_z.$$

4.002 θ = angle between **A** and **B**.

$$C = \sqrt{A^2 + B^2 + 2AB\cos\theta}.$$
$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB}.$$

4.003 If **a**, **b**, **c** are any three non-coplanar vectors of unit length, any vector, **R**, may be expressed:

$$\mathbf{R} = a\mathbf{a} + b\mathbf{b} + c\mathbf{c},$$

where a, b, c are the lengths of the projections of **R** upon **a**, **b**, **c** respectively.

4.004 Scalar product of two vectors:

$$S\mathbf{AB} = (\mathbf{AB}) = \mathbf{AB}$$

are equivalent notations.

$$\mathbf{AB} = AB\cos\widehat{AB}.$$

4.005 Vector product of two vectors:

$$V\mathbf{AB} = \mathbf{A} \times \mathbf{B} = [\mathbf{AB}] = \mathbf{C}.$$

C is a vector whose length is

$$C = AB\sin\widehat{AB}.$$

The direction of **C** is perpendicular to both **A** and **B** such that a right-handed rotation about **C** through the angle $\widehat{AB}$ turns **A** into **B**.

4.006 i, j, k are three unit vectors perpendicular to each other. If their directions coincide with the axes x, y, z of a rectangular system of coordinates:

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}.$$

4.007

$$\mathbf{ii} = \mathbf{i}^2 = \mathbf{jj} = \mathbf{j}^2 = \mathbf{kk} = \mathbf{k}^2 = 1,$$
$$\mathbf{ij} = \mathbf{ji} = \mathbf{jk} = \mathbf{kj} = \mathbf{ki} = \mathbf{ik} = 0.$$

4.008

$$V\mathbf{ij} = -V\mathbf{ji} = \mathbf{k},$$
$$V\mathbf{jk} = -V\mathbf{kj} = \mathbf{i},$$
$$V\mathbf{ki} = -V\mathbf{ik} = \mathbf{j}.$$

4.009

$$\mathbf{AB} = \mathbf{BA} = AB\cos\widehat{AB} = A_xB_x + A_yB_y + A_zB_z.$$

4.010

$$V\mathbf{AB} = -V\mathbf{BA} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
$$= (A_yB_z - A_zB_y)\mathbf{i} + (A_zB_x - A_xB_z)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k}.$$

4.10 If A, B, C, are any three vectors:

$$\mathbf{A}V\mathbf{BC} = \mathbf{B}V\mathbf{CA} = \mathbf{C}V\mathbf{AB}$$

= Volume of parallelepipedon having A, B, C as edges

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

4.11

1. $V\mathbf{A}(\mathbf{B}+\mathbf{C}) = V\mathbf{AB} + V\mathbf{AC}.$
2. $V(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D}) = V\mathbf{A}(\mathbf{C}+\mathbf{D}) + V\mathbf{B}(\mathbf{C}+\mathbf{D}).$
3. $V\mathbf{A}V\mathbf{BC} = \mathbf{B}S\mathbf{AC} - \mathbf{C}S\mathbf{AB}.$
4. $V\mathbf{A}V\mathbf{BC} + V\mathbf{B}V\mathbf{CA} + V\mathbf{C}V\mathbf{AB} = 0.$
5. $V\mathbf{AB}\cdot V\mathbf{CD} = \mathbf{AC}\cdot\mathbf{BD} - \mathbf{BC}\cdot\mathbf{AD}.$
6. $V(V\mathbf{AB}\cdot V\mathbf{CD}) = \mathbf{C}S(\mathbf{D}V\mathbf{AB}) - \mathbf{D}S(\mathbf{C}V\mathbf{AB})$
$= \mathbf{C}S(\mathbf{A}V\mathbf{BD}) - \mathbf{D}S(\mathbf{A}V\mathbf{BC})$
$= \mathbf{B}S(\mathbf{A}V\mathbf{CD}) - \mathbf{A}S(\mathbf{B}V\mathbf{CD})$
$= \mathbf{B}S(\mathbf{C}V\mathbf{DA}) - \mathbf{A}S(\mathbf{C}V\mathbf{DB}).$

4.20

1. $$dAB = \mathbf{A}d\mathbf{B} + \mathbf{B}d\mathbf{A}.$$
2. $$dV\mathbf{A}\mathbf{B} = V\mathbf{A}d\mathbf{B} + Vd\mathbf{A}\mathbf{B}$$ $$= V\mathbf{A}d\mathbf{B} - V\mathbf{B}d\mathbf{A}.$$

4.21

1. $$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$
2. $$\nabla\mathbf{A} = \operatorname{div}\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$
3. $$\nabla\phi = \operatorname{grad}\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}.$$
4. $$V\nabla\mathbf{A} = \operatorname{curl}\mathbf{A} = \operatorname{rot}\mathbf{A}$$ $$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$ $$= \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$$
5. $$\nabla\nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

4.22

1. $\operatorname{curl}\operatorname{grad}\phi = \operatorname{curl}\nabla\phi = V\nabla\nabla\phi = 0.$
2. $\operatorname{div}\operatorname{grad}\phi = \nabla\nabla\phi = \overline{\nabla}^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}.$
3. $\operatorname{div}\operatorname{curl}\mathbf{A} = 0.$
4. $\operatorname{curl}\operatorname{curl}\mathbf{A} = \operatorname{curl}^2\mathbf{A} = \nabla\operatorname{div}\mathbf{A} - \overline{\nabla}^2\mathbf{A}.$
5. $$\overline{\nabla}^2\mathbf{A} = \mathbf{i}\overline{\nabla}^2 A_x + \mathbf{j}\overline{\nabla}^2 A_y + \mathbf{k}\overline{\nabla}^2 A_z.$$
6. $$\mathbf{A}\nabla = A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z}.$$

4.23

1. $\nabla\mathbf{AB} = \text{grad}\,\mathbf{AB} = (\mathbf{A}\nabla)\mathbf{B} + (\mathbf{B}\nabla)\mathbf{A} + V\mathbf{A}\,\text{curl}\,\mathbf{B} + V\mathbf{B}\,\text{curl}\,\mathbf{A}.$
2. $\nabla V\mathbf{AB} = \text{div}\,V\mathbf{AB} = \mathbf{B}\,\text{curl}\,\mathbf{A} - \mathbf{A}\,\text{curl}\,\mathbf{B}.$
3. $V\nabla V\mathbf{AB} = (\mathbf{B}\nabla)\mathbf{A} - (\mathbf{A}\nabla)\mathbf{B} + \mathbf{A}\,\text{div}\,\mathbf{B} - \mathbf{B}\,\text{div}\,\mathbf{A}.$
4. $\text{div}\,\phi\mathbf{A} = \phi\,\text{div}\,\mathbf{A} + \mathbf{A}\nabla\phi.$
5. $\text{curl}\,\phi\mathbf{A} = V\cdot\nabla\phi\mathbf{A} + \phi\,\text{curl}\,\mathbf{A} = V\,\text{grad}\,\phi.\mathbf{A} + \phi\,\text{curl}\,\mathbf{A}.$
6. $\nabla\mathbf{A}^2 = 2(\mathbf{A}\nabla)\mathbf{A} + 2V\mathbf{A}\,\text{curl}\,\mathbf{A}.$
7. $\mathbf{C}(\mathbf{A}\nabla)\mathbf{B} = \mathbf{A}(\mathbf{C}\nabla)\mathbf{B} + \mathbf{A}V\mathbf{C}\,\text{curl}\,\mathbf{B}.$
8. $\mathbf{B}\nabla\mathbf{A}^2 = 2\mathbf{A}(\mathbf{B}\nabla)\mathbf{A}.$

4.24 R is a radius vector of length r and $\mathbf{r}$ a unit vector in the direction of R.

$$\mathbf{R} = r\mathbf{r},$$
$$r^2 = x^2 + y^2 + z^2.$$

1. $$\nabla\frac{1}{r} = -\frac{1}{r^3}\mathbf{R} = -\frac{1}{r^2}\mathbf{r}.$$
2. $$\nabla^2\frac{1}{r} = 0.$$
3. $$\nabla r = \frac{1}{r}\mathbf{R} = \mathbf{r} = \text{grad}\,r.$$
4. $$\nabla^2 r = \frac{2}{r}.$$
5. $$V\nabla\mathbf{R} = \text{curl}\,\mathbf{R} = 0.$$
6. $$\nabla\mathbf{R} = \text{div}\,\mathbf{R} = 3.$$
7. $$\frac{d\phi}{dr} = \mathbf{r}\nabla\phi.$$
8. $$(\mathbf{R}\nabla)\mathbf{A} = r\frac{d\mathbf{A}}{dr}.$$
9. $$(\mathbf{r}\nabla)\mathbf{A} = \frac{d\mathbf{A}}{dr}.$$
10. $$(\mathbf{A}\nabla)\mathbf{R} = \mathbf{A}.$$

4.30 $d\mathbf{S}$ = an element of area of a surface regarded as a vector whose direction is that of the positive normal to the surface.

ds = an element of arc of a curve regarded as a vector whose direction is that of the positive tangent to the curve.

4.31 Gauss's Theorem:

$$\iiint \operatorname{div} \mathbf{A} dV = \iint \mathbf{A} d\mathbf{S}.$$

4.32 Green's Theorem:

1. $\iiint \phi\nabla^2\psi dV + \iiint \nabla\phi\nabla\psi dV = \iint \phi\nabla\psi d\mathbf{S}$
2. $\iiint (\phi\nabla^2\psi - \psi\nabla^2\phi) dV = \iint (\phi\nabla\psi - \psi\nabla\phi) d\mathbf{S}.$

4.33 Stokes's Theorem:

$$\iint \operatorname{curl} \mathbf{A} d\mathbf{S} = \int \mathbf{A} d\mathbf{s}.$$

4.40 A polar vector is one whose components, referred to a rectangular system of axes, all change in sign when the three axes are reversed.

4.401 An axial vector is one whose components are unchanged when the axes are reversed.

4.402 The vector product of two polar or of two axial vectors is an axial vector.

4.403 The vector product of a polar and an axial vector is a polar vector.

4.404 The curl of a polar vector is an axial vector and the curl of an axial vector is a polar vector.

4.405 The scalar product of two polar or of two axial vectors is a true scalar, i.e., it keeps its sign if the axes to which the vectors are referred are reversed.

4.406 The scalar product of an axial vector and a polar vector is a pseudo-scalar, i.e., it changes in sign when the axes of reference are reversed.

4.407 The product or quotient of a polar vector and a true scalar is a polar vector; of an axial vector and a true scalar an axial vector; of a polar vector and a pseudo-scalar an axial vector; of an axial vector and a pseudo-scalar a polar vector.

4.408 The gradient of a true scalar is a polar vector; the gradient of a pseudo-scalar is an axial vector.

4.409 The divergence of a polar vector is a true scalar; of an axial vector a pseudo-scalar.

4.6 Linear Vector Functions.

4.610 A vector $\mathbf{Q}$ is a linear vector function of a vector $\mathbf{R}$ if its components, Q_1, Q_2, Q_3, along any three non-coplanar axes are linear functions of the components R_1, R_2, R_3 of $\mathbf{R}$ along the same axes.

4.611 Linear Vector Operator. If $\hat{\omega}$ is the linear vector operator,

$$\mathbf{Q} = \hat{\omega}\mathbf{R}.$$

This is equivalent to the three scalar equations,

$$Q_1 = \omega_{11}R_1 + \omega_{12}R_2 + \omega_{13}R_3,$$
$$Q_2 = \omega_{21}R_1 + \omega_{22}R_2 + \omega_{23}R_3,$$
$$Q_3 = \omega_{31}R_1 + \omega_{32}R_2 + \omega_{33}R_3.$$

4.612 If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are the three non-coplanar unit axes,

$$\omega_{11} = S.\mathbf{a}\hat{\omega}\mathbf{a}, \quad \omega_{21} = S.\mathbf{b}\hat{\omega}\mathbf{a}, \quad \omega_{31} = S.\mathbf{c}\hat{\omega}\mathbf{a},$$
$$\omega_{12} = S.\mathbf{a}\hat{\omega}\mathbf{b}, \quad \omega_{22} = S.\mathbf{b}\hat{\omega}\mathbf{b}, \quad \omega_{32} = S.\mathbf{c}\hat{\omega}\mathbf{b},$$
$$\omega_{13} = S.\mathbf{a}\hat{\omega}\mathbf{c}, \quad \omega_{23} = S.\mathbf{b}\hat{\omega}\mathbf{c} \quad \omega_{33} = S.\mathbf{c}\hat{\omega}\mathbf{c}.$$

4.613 The conjugate linear vector operator $\hat{\omega}'$ is obtained from $\hat{\omega}$ by replacing ω_{hk} by ω_{kh}; $h, k = 1, 2, 3$.

4.614 In the symmetrical, or self-conjugate linear vector operator, denoted by ω,

$$\omega = \tfrac{1}{2}(\hat{\omega} + \hat{\omega}').$$

Hence by **4.612**

$$S.\mathbf{a}\omega\mathbf{b} = S.\mathbf{b}\omega\mathbf{a}, \text{ etc.}$$

4.615 The general linear vector function $\hat{\omega}\mathbf{R}$ may always be resolved into the sum of a self-conjugate linear vector function of $\mathbf{R}$ and the vector product of $\mathbf{R}$ by a vector $\mathbf{c}$:

$$\hat{\omega}\mathbf{R} = \omega\mathbf{R} + V.\mathbf{c}\mathbf{R},$$

where

$$\omega = \tfrac{1}{2}(\hat{\omega} + \hat{\omega}'),$$

and

$$\mathbf{c} = \tfrac{1}{2}(\omega_{32} - \omega_{23})\mathbf{i} + \tfrac{1}{2}(\omega_{13} - \omega_{31})\mathbf{j} + \tfrac{1}{2}(\omega_{21} - \omega_{12})\mathbf{k},$$

if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are three mutually perpendicular unit vectors.

4.616 The general linear vector operator $\hat{\omega}$ may be determined by three non-

$$A = a\omega_{11} + b\omega_{12} + c\omega_{13},$$
$$B = a\omega_{21} + b\omega_{22} + c\omega_{23},$$
$$C = a\omega_{31} + b\omega_{32} + c\omega_{33},$$

and

$$\hat{\omega} = aS.A + bS.B + cS.C.$$

4.617 If $\hat{\omega}$ is the general linear vector operator and $\hat{\omega}'$ its conjugate,

$$\hat{\omega}R = R\hat{\omega}',$$
$$\hat{\omega}'R = R\hat{\omega}$$

4.620 The symmetrical or self-conjugate linear vector operator has three mutually perpendicular axes. If these be taken along i, j, k,

$$\omega = iS.\omega_1 i + jS.\omega_2 j + kS.\omega_3 k,$$

where ω_1, ω_2, ω_3 are scalar quantities, the principal values of ω.

4.621 Referred to any system of three mutually perpendicular unit vectors, a, b, c, the self-conjugate operator, ω, is determined by the three vectors (**4.616**):

$$A = \omega a = a\omega_{11} + b\omega_{12} + c\omega_{13},$$
$$B = \omega b = a\omega_{21} + b\omega_{22} + c\omega_{23},$$
$$C = \omega c = a\omega_{31} + b\omega_{32} + c\omega_{33},$$

where

$$\omega_{hk} = \omega_{kh},$$
$$\omega = aS.A + bS.B + cS.C.$$

4.622 If n is one of the principal values, ω_1, ω_2, ω_3, these are given by the roots of the cubic,

$$n^3 - n^2(S.Aa + S.Bb + S.Cc) + n(S.aVBC + S.bVCA + S.cVAB) - S.AVBC = 0.$$

4.623 In transforming from one to another system of rectangular axes the following are invariant:

$$S.Aa + S.Bb + S.Cc = \omega_1 + \omega_2 + \omega_3.$$
$$SaVBC + S.bVCA + S.cVAB = \omega_2\omega_3 + \omega_3\omega_1 + \omega_1\omega_2.$$
$$S.AVBC = \omega_1\omega_2\omega_3.$$

4.624

$$\omega_1 + \omega_2 + \omega_3 = \omega_{11} + \omega_{22} + \omega_{33},$$
$$\omega_2\omega_3 + \omega_3\omega_1 + \omega_1\omega_2 = \omega_{22}\omega_{33} + \omega_{33}\omega_{11} + \omega_{11}\omega_{22} - \omega^2_{23} - \omega^2_{31} + \omega^2_{12},$$
$$\omega_1\omega_2\omega_3 = \omega_{11}\omega_{22}\omega_{33} + 2\omega_{23}\omega_{31}\omega_{12} - \omega_{11}\omega^2_{23} - \omega_{22}\omega^2_{31} - \omega_{33}\omega^2_{12}.$$

4.625 The principal axes of the self-conjugate operator, ω, are those of the quadric:

$$\omega_{11}x^2 + \omega_{22}y^2 + \omega_{33}z^2 + 2\omega_{23}yz + 2\omega_{31}zx + 2\omega_{12}xy = \text{const.},$$

... are rectangular axes in the direction of a, b, c respectively.

4.626 Referred to its principal axes the equation of the quadric is,

$$\omega_1 x^2 + \omega_2 y^2 + \omega_3 z^2 = \text{const.}$$

4.627 Applying the self-conjugate operator, ω, successively,

$$\omega\mathbf{R} = \mathbf{i}\omega_1 R_1 + \mathbf{j}\omega_2 R_2 + \mathbf{k}\omega_3 R_3.$$

$$\omega\omega\mathbf{R} = \omega^2\mathbf{R} = \mathbf{i}\omega_1^2 R_1 + \mathbf{j}\omega_2^2 R_2 + \mathbf{k}\omega_3^2 R_3.$$

$$\omega\omega^2\mathbf{R} = \omega^3\mathbf{R} = \mathbf{i}\omega_1^3 R_1 + \mathbf{j}\omega_2^3 R_2 + \mathbf{k}\omega_3^3 R_3.$$

$$\cdots$$

$$\cdots$$

$$\omega^{-1}\mathbf{R} = \mathbf{i}\frac{R_1}{\omega_1} + \mathbf{j}\frac{R_2}{\omega_2} + \mathbf{k}\frac{R_3}{\omega_3}.$$

$$\cdots$$

$$\cdots$$

4.628 Applying a number of self-conjugate operators, α, β, ..., all with the same axes but with different principal values $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$, ...

$$\alpha\mathbf{R} = \mathbf{i}\alpha_1 R_1 + \mathbf{j}\alpha_2 R_2 + \mathbf{k}\alpha_3 R_3.$$

$$\beta\alpha\mathbf{R} = \alpha\beta\mathbf{R} = \mathbf{i}\alpha_1\beta_1 R_1 + \mathbf{j}\alpha_2\beta_2 R_2 + \mathbf{k}\alpha_3\beta_3 R_3.$$

$$\cdots$$

4.629

$$S.\mathbf{Q}\omega\mathbf{R} = S.\mathbf{R}\omega\mathbf{Q}$$
$$= \omega_1 Q_1 R_1 + \omega_2 Q_2 R_2 + \omega_3 Q_3 R_3.$$

V. CURVILINEAR COÖRDINATES

5.00 Given three surfaces,

$$1. \quad \begin{cases} u = f_1(x, y, z), \\ v = f_2(x, y, z), \\ w = f_3(x, y, z). \end{cases}$$

$$2. \quad \begin{cases} x = \phi_1(u, v, w), \\ y = \phi_2(u, v, w), \\ z = \phi_3(u, v, w). \end{cases}$$

$$3. \quad \begin{cases} \dfrac{1}{h_1^2} = \left(\dfrac{\partial\phi_1}{\partial u}\right)^2 + \left(\dfrac{\partial\phi_2}{\partial u}\right)^2 + \left(\dfrac{\partial\phi_3}{\partial u}\right)^2, \\ \dfrac{1}{h_2^2} = \left(\dfrac{\partial\phi_1}{\partial v}\right)^2 + \left(\dfrac{\partial\phi_2}{\partial v}\right)^2 + \left(\dfrac{\partial\phi_3}{\partial v}\right)^2, \\ \dfrac{1}{h_3^2} = \left(\dfrac{\partial\phi_1}{\partial w}\right)^2 + \left(\dfrac{\partial\phi_2}{\partial w}\right)^2 + \left(\dfrac{\partial\phi_3}{\partial w}\right)^2. \end{cases}$$

$$4. \quad \begin{cases} g_1 = \dfrac{\partial\phi_1}{\partial v}\dfrac{\partial\phi_1}{\partial w} + \dfrac{\partial\phi_2}{\partial v}\dfrac{\partial\phi_2}{\partial w} + \dfrac{\partial\phi_3}{\partial v}\dfrac{\partial\phi_3}{\partial w}, \\ g_2 = \dfrac{\partial\phi_1}{\partial w}\dfrac{\partial\phi_1}{\partial u} + \dfrac{\partial\phi_2}{\partial w}\dfrac{\partial\phi_2}{\partial u} + \dfrac{\partial\phi_3}{\partial w}\dfrac{\partial\phi_3}{\partial u}, \\ g_3 = \dfrac{\partial\phi_1}{\partial u}\dfrac{\partial\phi_1}{\partial v} + \dfrac{\partial\phi_2}{\partial u}\dfrac{\partial\phi_2}{\partial v} + \dfrac{\partial\phi_3}{\partial u}\dfrac{\partial\phi_3}{\partial v}. \end{cases}$$

5.01 The linear element of arc, ds, is given by:

$$ds^2 = dx^2 + dy^2 + dz^2 = \frac{du^2}{h_1^2} + \frac{dv^2}{h_2^2} + \frac{dw^2}{h_3^2} + 2g_1\,dv\,dw + 2g_2\,dw\,du + 2g_3\,du\,dv.$$

5.02 The surface elements, areas of parallelograms on the three surfaces, are:

$$dS_u = \frac{dv\,dw}{h_2h_3}\sqrt{1 - h_2^2h_3^2g_1^2},$$

$$dS_v = \frac{dw\,du}{h_3h_1}\sqrt{1 - h_3^2h_1^2g_2^2},$$

$$dS_w = \frac{du\,dv}{h_1h_2}\sqrt{1 - h_1^2h_2^2g_3^2}.$$

5.07 A vector, A, will have three components in the directions of the normals to the orthogonal surfaces u, v, w:

$$A = \sqrt{A_u^2 + A_v^2 + A_w^2}.$$

5.08

1. $$\operatorname{div} \mathbf{A} = h_1h_2h_3 \left\{ \frac{\partial}{\partial u}\left(\frac{A_u}{h_2h_3}\right) + \frac{\partial}{\partial v}\left(\frac{A_v}{h_3h_1}\right) + \frac{\partial}{\partial w}\left(\frac{A_w}{h_1h_2}\right) \right\}.$$

2. $$\nabla^2 = h_1h_2h_3 \left\{ \frac{\partial}{\partial u}\left(\frac{h_1}{h_2h_3}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_2}{h_3h_1}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_3}{h_1h_2}\frac{\partial}{\partial w}\right) \right\}.$$

3. $$\begin{cases} \operatorname{curl}_u \mathbf{A} = h_2h_3 \left\{ \dfrac{\partial}{\partial v}\left(\dfrac{A_w}{h_3}\right) - \dfrac{\partial}{\partial w}\left(\dfrac{A_v}{h_2}\right) \right\}, \\ \operatorname{curl}_v \mathbf{A} = h_3h_1 \left\{ \dfrac{\partial}{\partial w}\left(\dfrac{A_u}{h_1}\right) - \dfrac{\partial}{\partial u}\left(\dfrac{A_w}{h_3}\right) \right\}, \\ \operatorname{curl}_w \mathbf{A} = h_1h_2 \left\{ \dfrac{\partial}{\partial u}\left(\dfrac{A_v}{h_2}\right) - \dfrac{\partial}{\partial v}\left(\dfrac{A_u}{h_1}\right) \right\}. \end{cases}$$

5.09 The gradient of a scalar function, ψ, has three components in the directions of the normals to the three orthogonal surfaces:

$$h_1\frac{\partial\psi}{\partial u},\ h_2\frac{\partial\psi}{\partial v},\ h_3\frac{\partial\psi}{\partial w}.$$

5.20 Spherical Polar Coördinates.

1. $$\begin{cases} u = r, \\ v = \theta, \\ w = \phi. \end{cases}$$

2. $$\begin{cases} x = r\sin\theta\cos\phi, \\ y = r\sin\theta\sin\phi, \\ z = r\cos\theta. \end{cases}$$

3. $$h_1 = 1,\ h_2 = \frac{1}{r},\ h_3 = \frac{1}{r\sin\theta}$$

4. $$\begin{cases} dS_r = r^2\sin\theta\, d\theta\, d\phi, \\ dS_\theta = r\sin\theta\, dr\, d\phi, \\ dS_\phi = r\, dr\, d\theta. \end{cases}$$

5. $$d\tau = r^2\sin\theta\, dr\, d\theta\, d\phi.$$

6. $$\operatorname{div}\mathbf{A} = \frac{1}{r^2\sin\theta}\left\{ \sin\theta\frac{\partial}{\partial r}\left(r^2A_r\right) + r\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + r\frac{\partial A_\phi}{\partial\phi} \right\}.$$

7. $$\nabla^2 = \frac{1}{r^2\sin\theta}\left\{ \sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2} \right\}.$$

2.
$$\begin{cases} x^2 = \dfrac{(a^2+u)\,(a^2+v)\,(a^2+w)}{(a^2-b^2)\,(a^2-c^2)}, \\[2mm] y^2 = -\dfrac{(b^2+u)\,(b^2+v)\,(b^2+w)}{(b^2-c^2)\,(a^2-b^2)}, \\[2mm] z^2 = \dfrac{(c^2+u)\,(c^2+v)\,(c^2+w)}{(a^2-c^2)\,(b^2-c^2)}. \end{cases}$$

3.
$$\begin{cases} h_1^2 = \dfrac{4(a^2+u)\,(b^2+u)\,(c^2+u)}{(u-v)\,(u-w)}, \\[2mm] h_2^2 = \dfrac{4(a^2+v)\,(b^2+v)\,(c^2+v)}{(v-w)\,(v-u)}, \\[2mm] h_3^2 = \dfrac{4(a^2+w)\,(b^2+w)\,(c^2+w)}{(w-u)\,(w-v)}. \end{cases}$$

4.
$$\begin{aligned} \operatorname{div} \mathbf{A} = {} & 2\,\frac{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}{(u-v)\,(u-w)}\,\frac{\partial}{\partial u}\left(\sqrt{(u-v)\,(u-w)}\,A_u\right) \\ & + 2\,\frac{\sqrt{(a^2+v)\,(b^2+v)\,(c^2+v)}}{(v-w)\,(u-v)}\,\frac{\partial}{\partial v}\left(\sqrt{(w-v)\,(u-v)}\,A_v\right) \\ & + 2\,\frac{\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}}{(u-w)\,(v-w)}\,\frac{\partial}{\partial w}\left(\sqrt{(u-w)\,(v-w)}\,A_w\right). \end{aligned}$$

5.
$$\begin{aligned} \nabla^2 = {} & 4\,\frac{\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}}{(u-v)\,(u-w)}\,\frac{\partial}{\partial u}\left(\sqrt{(a^2+u)\,(b^2+u)\,(c^2+u)}\,\frac{\partial}{\partial u}\right) \\ & + 4\,\frac{\sqrt{(a^2+v)\,(c^2+v)\,(b^2+v)}}{(u-v)\,(v-w)}\,\frac{\partial}{\partial v}\left(\sqrt{(a^2+v)\,(b^2+v)\,(c^2+v)}\,\frac{\partial}{\partial v}\right) \\ & + 4\,\frac{\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}}{(u-w)\,(v-w)}\,\frac{\partial}{\partial w}\left(\sqrt{(a^2+w)\,(b^2+w)\,(c^2+w)}\,\frac{\partial}{\delta w}\right). \end{aligned}$$

$$\begin{aligned} \operatorname{curl}_u \mathbf{A} = \frac{2}{v-w}\Bigg\{ & \sqrt{\frac{(a^2+v)\,(b^2+v)\,(c^2+v)}{u-v}}\,\frac{\partial}{\partial v}\left(\sqrt{w-v}\,A_w\right) \\ & - \sqrt{\frac{(a^2+w)\,(b^2+w)\,(c^2+w)}{u-w}}\,\frac{\partial}{\partial w}\left(\sqrt{v-w}\,A_v\right\}. \end{aligned}$$

$$\begin{aligned} \operatorname{curl}_v \mathbf{A} = \frac{2}{u-w}\Bigg\{ & \sqrt{\frac{(a^2+w)\,(b^2+w)\,(c^2+w)}{v-w}}\,\frac{\partial}{\partial w}\left(\sqrt{u-w}\,A_u\right) \\ & - \sqrt{\frac{(a^2+u)\,(b^2+u)\,(c^2+u)}{v-u}}\,\frac{\partial}{\partial u}\left(\sqrt{w-u}\,A_w\right)\Bigg\} \end{aligned}$$

$$\begin{aligned} \operatorname{curl}_w \mathbf{A} = \frac{2}{u-v}\Bigg\{ & \sqrt{\frac{(a^2+u)\,(b^2+u)\,(c^2+u)}{w-u}}\,\frac{\partial}{\partial u}\left(\sqrt{v-u}\,A_v\right) \\ & - \sqrt{\frac{(a^2+v)\,(b^2+v)\,(c^2+v)}{w-v}}\,\frac{\partial}{\partial v}\left(\sqrt{u-v}\,A_u\right)\Bigg\}. \end{aligned}$$

5.23 Conical Coördinates.

The three orthogonal surfaces are: the spheres,

1. $$x^2 + y^2 + z^2 = u^2,$$

the two cones:

2. $$\frac{x^2}{v^2} + \frac{y^2}{v^2 - b^2} + \frac{z^2}{v^2 - c^2} = 0,$$

3. $$\frac{x^2}{w^2} + \frac{y^2}{w^2 - b^2} + \frac{z^2}{w^2 - c^2} = 0,$$

$$c^2 > v^2 > b^2 > w^2.$$

4. $$\begin{cases} x^2 = \dfrac{u^2 v^2 w^2}{b^2 c^2}, \\ y^2 = \dfrac{u^2 (v^2 - b^2)(w^2 - b^2)}{b^2 (b^2 - c^2)}, \\ z^2 = \dfrac{u^2 (v^2 - c^2)(w^2 - c^2)}{c^2 (c^2 - b^2)}. \end{cases}$$

5. $$h_1 = 1, \quad h_2^2 = \frac{(v^2 - b^2)(c^2 - v^2)}{u^2 (v^2 - w^2)}, \quad h_3^2 = \frac{(b^2 - w^2)(c^2 - w^2)}{u^2 (v^2 - w^2)}.$$

6. $$\text{div } A = \frac{1}{u^2}\frac{\partial}{\partial u}(u^2 A_u) + \frac{\sqrt{(v^2 - b^2)(c^2 - v^2)}}{u(v^2 - w^2)}\frac{\partial}{\partial v}\left(\sqrt{v^2 - w^2}\, A_v\right)$$
$$+ \frac{\sqrt{(b^2 - w^2)(c^2 - w^2)}}{u(v^2 - w^2)}\frac{\partial}{\partial w}\left(\sqrt{v^2 - w^2}\, A_w\right).$$

7. $$\nabla^2 = \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 \frac{\partial}{\partial u}\right) + \frac{\sqrt{(v^2 - b^2)(c^2 - v^2)}}{u^2 (v^2 - w^2)}\frac{\partial}{\partial v}\left(\sqrt{(v^2 - b^2)(c^2 - v^2)}\frac{\partial}{\partial v}\right)$$
$$+ \frac{\sqrt{(b^2 - w^2)(c^2 - w^2)}}{u^2 (v^2 - w^2)}\frac{\partial}{\partial w}\left(\sqrt{(b^2 - w^2)(c^2 - w^2)}\frac{\partial}{\partial w}\right).$$

8. $$\begin{cases} \text{curl}_u A = \dfrac{1}{u(v^2 - w^2)}\left\{\sqrt{(v^2 - b^2)(c^2 - v^2)}\dfrac{\partial}{\partial v}\left(\sqrt{v^2 - w^2}\, A_w\right)\right. \\ \qquad\qquad \left. - \sqrt{(b^2 - w^2)(c^2 - w^2)}\dfrac{\partial}{\partial w}\left(\sqrt{v^2 - w^2}\, A_v\right)\right\}, \\ \text{curl}_v A = \dfrac{\sqrt{(b^2 - w^2)(c^2 - w^2)}}{u\sqrt{v^2 - w^2}}\dfrac{\partial A_u}{\partial w} - \dfrac{1}{u}\dfrac{\partial}{\partial u}\left(u A_w\right), \\ \text{curl}_w A = \dfrac{1}{u}\dfrac{\partial}{\partial u}\left(u A_v\right) - \dfrac{\sqrt{(v^2 - b^2)(c^2 - v^2)}}{u\sqrt{v^2 - w^2}}\dfrac{\partial A_u}{\partial v}. \end{cases}$$

5.30 Elliptic Cylinder Coördinates.

The three orthogonal surfaces are:

1. The elliptic cylinders:

$$\frac{x^2}{c^2 u^2} + \frac{y^2}{c^2 (u^2 - 1)} = 1,$$

2. The hyperbolic cylinders:

$$\frac{x^2}{c^2v^2} - \frac{y^2}{c^2(1 - v^2)} = 1.$$

3. The planes: $z = w.$

$2c$ is the distance between the foci of the confocal ellipses and hyperbolas:

4. $$x = cuv.$$

5. $$y = c\sqrt{u^2 - 1}\,\sqrt{1 - v^2}.$$

6. $$\frac{1}{h_1^2} = \frac{1}{h_2^2} = c^2(u^2 - v^2), \quad h_3 = 1.$$

7. $$\operatorname{div} \mathbf{A} = \frac{1}{c(u^2 - v^2)} \left\{ \frac{\partial}{\partial u}\left(\sqrt{u^2 - v^2}\, A_u\right) + \frac{\partial}{\partial v}\left(\sqrt{u^2 - v^2}\, A_v\right) \right\} + \frac{\partial A_z}{\partial z}.$$

8. $$\nabla^2 = \frac{1}{c^2(u^2 - v^2)}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + \frac{\partial^2}{\partial z^2}.$$

9. $$\begin{cases} \operatorname{curl}_u \mathbf{A} = \dfrac{1}{c\sqrt{u^2 - v^2}}\dfrac{\partial A_z}{\partial v} - \dfrac{\partial A_v}{\partial z}, \\ \operatorname{curl}_v \mathbf{A} = \dfrac{\partial A_u}{\partial z} - \dfrac{1}{c\sqrt{u^2 - v^2}}\dfrac{\partial A_z}{\partial u}, \\ \operatorname{curl}_z \mathbf{A} = \dfrac{1}{c(u^2 - v^2)}\left\{\dfrac{\partial}{\partial u}\left(\sqrt{u^2 - v^2}\, A_v\right) - \dfrac{\partial}{\partial v}\left(\sqrt{u^2 - v^2}\, A_u\right)\right\}. \end{cases}$$

5.31 Parabolic Cylinder Coördinates.

The three orthogonal surfaces are the two parabolic cylinders:

1. $$y^2 = 4cux + 4c^2u^2.$$

2. $$y^2 = -4cvx + 4c^2v^2.$$

And the planes:

3. $$z = w.$$

4. $$x = c(v - u).$$

5. $$y = 2c\sqrt{uv}.$$

6. $$\frac{1}{h_1^2} = \frac{u + v}{u}, \quad \frac{1}{h_2^2} = \frac{u + v}{v}, \quad h_3 = 1.$$

7. $$\operatorname{div} \mathbf{A} = \frac{\sqrt{uv}}{u + v} \left\{ \frac{\partial}{\partial u}\left(\sqrt{\frac{u + v}{v}}\, A_u\right) + \frac{\partial}{\partial v}\left(\sqrt{\frac{u + v}{u}}\, A_v\right) \right\} + \frac{\partial A_z}{\partial z}.$$

8. $$\nabla^2 = \frac{\sqrt{uv}}{u + v}\left\{\frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{v}{u}\frac{\partial}{\partial v}\right)\right\} + \frac{\partial^2}{\partial z^2}.$$

9. $$\begin{cases} \operatorname{curl}_u \mathrm{A} = \sqrt{\dfrac{v}{u+v}}\,\dfrac{\partial A_z}{\partial v} - \dfrac{v}{u+v}\,\dfrac{\partial A_v}{\partial z}, \\ \operatorname{curl}_v \mathrm{A} = \dfrac{u}{u+v}\,\dfrac{\partial A_u}{\partial z} - \sqrt{\dfrac{u}{u+v}}\,\dfrac{\partial A_z}{\partial u}, \\ \operatorname{curl}_z \mathrm{A} = \dfrac{\sqrt{uv}}{u+v}\left\{\dfrac{\partial}{\partial u}\left(\sqrt{\dfrac{v}{u+v}}\,A_v\right) - \dfrac{\partial}{\partial v}\left(\sqrt{\dfrac{u}{u+v}}\,A_u\right)\right\}. \end{cases}$$

5.40 Helical Coördinates. (Nicholson, Phil. Mag. 19, 77, 1910.)

A cylinder of any cross-section is wound on a circular cylinder in the form of a helix of angle α. a = radius of circular cylinder on which the central line of the normal cross-sections of the helical cylinder lies. The z axis is along the axis of the cylinder of radius a.

$u = \rho$ and $v = \phi$ are the polar coördinates in the plane of any normal section of the helical cylinder. ϕ is measured from a line perpendicular to z and to the tangent to the cylinder.

$w = \theta$ = the twist in a plane perpendicular to z of the radius in that plane measured from a line parallel to the x axis:

1. $$\begin{cases} x = (a + \rho\cos\phi)\cos\theta + \rho\sin\alpha\sin\theta\sin\phi, \\ y = (a + \rho\cos\phi)\sin\theta - \rho\sin\alpha\cos\theta\sin\phi, \\ z = a\,\theta\tan\alpha + \rho\cos\alpha\sin\phi. \end{cases}$$

2. $$\begin{cases} h_1 = 1, \quad h_2 = \dfrac{1}{\rho} \\ h_3^2 = \dfrac{1}{a^2\sec^2\alpha + 2a\rho\cos\phi + \rho^2(\cos^2\phi + \sin^2\alpha\sin^2\phi)}. \end{cases}$$

5.50 Surfaces of Revolution.

z-axis = axis of revolution.

ρ, θ = polar coördinates in any plane perpendicular to z-axis.

1. $$ds^2 = dz^2 + d\rho^2 + \rho^2 d\theta^2$$
$$= \frac{du^2}{h_1^2} + \frac{dv^2}{h_2^2} + \frac{dw^2}{h_3^2}.$$

In any meridian plane, z, ρ, determine u, v, from:

2. $$f(z + i\rho) = u + iv.$$

3. $$w = \theta.$$

5.51 Spheroidal Coördinates (Prolate Spheroids):

1. $$z + i\rho = c \cosh (u + iv).$$

2. $$\begin{cases} z = c \cosh u \cos v, \\ \rho = c \sinh u \sin v. \end{cases}$$

The three orthogonal surfaces are the ellipsoids and hyperboloids of revolution, and the planes, θ:

3. $$\begin{cases} \dfrac{z^2}{c^2 \cosh^2 u} + \dfrac{\rho^2}{c^2 \sinh^2 u} = 1, \\ \dfrac{z^2}{c^2 \cos^2 v} - \dfrac{\rho^2}{c^2 \sin^2 v} = 1. \end{cases}$$

With $\cos u = \lambda$, $\cos v = \mu$:

4. $$\begin{cases} z = c\,\lambda\,\mu, \\ \rho = c\sqrt{(\lambda^2 - 1)\,(1 - \mu^2)}. \end{cases}$$

5. $$h_1^2 = \frac{\lambda^2 - 1}{c^2(\lambda^2 - \mu^2)}, \quad h_2^2 = \frac{1 - \mu^2}{c^2\,(\lambda^2 - \mu^2)}, \quad h_3^2 = \frac{1}{c^2(\lambda^2 - 1)\,(1 - \mu^2)}$$

5.52 Spheroidal Coördinates (Oblate Spheroids):

1. $$\rho + iz = c \cosh(u + iv).$$

2. $$\begin{aligned} z &= c \sinh u \sin v. \\ \rho &= c \cosh u \cos v. \end{aligned}$$

3. $$\cosh u = \lambda, \quad \cos v = \mu.$$

4. $$h_1^2 = \frac{1 - \mu^2}{c^2(\lambda^2 - \mu^2)}, \quad h_2^2 = \frac{\lambda^2 - 1}{c^2(\lambda^2 - \mu^2)}, \quad h_3^2 = \frac{1}{c^2(\lambda^2 - 1)\,(1 - \mu^2)}.$$

5.53 Parabolic Coördinates:

1. $$z + i\rho = c(u + iv)^2.$$

2. $$\begin{cases} z = c(u^2 - v^2), \\ \rho = 2cuv. \end{cases}$$

3. $$u^2 = \lambda, \quad v^2 = \mu.$$

4. $$h_1 = \frac{1}{c}\sqrt{\frac{\lambda}{\lambda+\mu}}, \quad h_2 = \frac{1}{c}\sqrt{\frac{\mu}{\lambda+\mu}}, \quad h_3 = \frac{1}{2c\sqrt{\lambda\mu}}.$$

5.54 Toroidal Coördinates:

1. $$u + iv = \log\frac{z + a + i\rho}{z - a + i\rho},$$

$$\rho = \frac{a \sinh u}{\cosh u - \cos v},$$

2. $$z = \frac{a \sin v}{\cosh u - \cos v}.$$

3. $$h_1 = h_2 = \frac{\cosh u - \cos v}{a}, \quad h_3 = \frac{\cosh u - \cos v}{a \sinh u}.$$

The three orthogonal surfaces are:

(a) Anchor rings, whose axial circles have radii,

$$a \coth u,$$

and whose cross-sections are circles of radii,

$$a \operatorname{csch} u;$$

(b) Spheres, whose centers are on the axis of revolution at distances,

$$\pm a \cot v,$$

from the origin, whose radii are,

$$a \csc v,$$

and which accordingly have a common circle,

$$\rho = a, \; z = 0;$$

(c) Planes through the axis,

$$w = \theta = \text{const.}$$

VI. INFINITE SERIES

6.00 An infinite series:

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots .$$

is absolutely convergent if the series formed of the moduli of its terms:

$$| u_1 | + | u_2 | + | u_3 | + \ldots .$$

is convergent.

A series which is convergent, but whose moduli do not form a convergent series, is conditionally convergent.

TESTS FOR CONVERGENCE

6.011 Comparison test. The series Σu_n is absolutely convergent if $| u_n |$ is less than $C | v_n |$ where C is a number independent of n, and v_n is the nth term of another series which is known to be absolutely convergent.

6.012 Cauchy's test. If

$$\operatorname*{Limit}_{n \to \infty} | u_n |^{\frac{1}{n}} < 1,$$

the series Σu_n is absolutely convergent.

6.013 D'Alembert's test. If for all values of n greater than some fixed value, r, the ratio $\left| \frac{u_{n+1}}{u_n} \right|$ is less than ρ, where ρ is a positive number less than unity and independent of n, the series Σu_n is absolutely convergent.

6.014 Cauchy's integral test. Let $f(x)$ be a steadily decreasing positive function such that,

$$f(n) \geqq a_n.$$

Then the positive term series Σa_n is convergent if,

$$\int_m^{\infty} f(x)dx,$$

is convergent.

6.015 Raabe's test. The positive term series Σa_n is convergent if,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geqq l \quad \text{where } l > 1.$$

It is divergent if,

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leqq 1.$$

6.020 Alternating series. A series of real terms, alternately positive and negative, is convergent if $u_{n+1} \leqq u_n$ and

$$\lim_{n \to \infty} u_n = 0.$$

In such a series the sum of the first s terms differs from the sum of the series by a quantity less than the numerical value of the $(s+1)$st term.

6.025 If $\lim\limits_{n \to \infty} \left| \dfrac{u_{n+1}}{u_n} \right| = 1$, the series Σu_n will be absolutely convergent if there is a positive number c, independent of n, such that,

$$\lim_{n \to \infty} n \left\{ \left| \frac{u_{n+1}}{u_n} \right| - 1 \right\} = -1 - c.$$

6.030 The sum of an absolutely convergent series is not affected by changing the order in which the terms occur.

6.031 Two absolutely convergent series,

$$S = u_1 + u_2 + u_3 + \ldots \ldots$$
$$T = v_1 + v_2 + v_3 + \ldots \ldots$$

may be multiplied together, and the sum of the products of their terms, written in any order, is ST,

$$ST = u_1v_1 + u_2v_1 + u_1v_2 + \ldots \ldots$$

6.032 An absolutely convergent power series may be differentiated or integrated term by term and the resulting series will be absolutely convergent and equal to the differential or integral of the sum of the given series.

6.040 Uniform Convergence. An infinite series of functions of x,

$$S(x) = u_1(x) + u_2(x) + u_3(x) + \ldots \ldots$$

is uniformly convergent within a certain region of the variable x if a finite number, N, can be found such that for all values of $n > N$ the absolute value of the remainder, $|R_n|$ after n terms is less than an assigned arbitrary small quantity ϵ at all points within the given range.

Example. The series,

6.041 A uniformly convergent series is not necessarily absolutely convergent, nor is an absolutely convergent series necessarily uniformly convergent.

6.042 A sufficient, though not necessary, test for uniform convergence is as follows:

If for all values of x within a certain region the moduli of the terms of the series,

$$S = u_1(x) + u_2(x) + \ldots \ldots,$$

are less than the corresponding terms of a convergent series of positive terms,

$$T = M_1 + M_2 + M_3 + \ldots \ldots,$$

where M_n is independent of x, then the series S is uniformly convergent in the given region.

6.043 A power series is uniformly convergent at all points within its circle of convergence.

6.044 A uniformly convergent series,

$$S = u_1(x) + u_2(x) + \ldots \ldots,$$

may be integrated term by term, and,

$$\int S\,dx = \sum_{n=1}^{\infty} \int u_n(x)\,dx.$$

6.045 A uniformly convergent series,

$$S = u_1(x) + u_2(x) + \ldots \ldots,$$

may be differentiated term by term, and if the resulting series is uniformly convergent,

$$\frac{d}{dx} S = \sum_{n=1}^{\infty} \frac{d}{dx} u_n(x).$$

6.100 Taylor's theorem.

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots \ldots + \frac{h^n}{n!} f^{(n)}(x) + R_n.$$

6.101 Lagrange's form for the remainder:

$$R_n = f^{(n+1)}(x + \theta h) \cdot \frac{h^{n+1}}{(n+1)!}; \quad 0 < \theta < 1.$$

6.102 Cauchy's form for the remainder:

$$R_n = f^{(n+1)}(x + \theta h) \frac{h^{n+1}(1-\theta)^n}{n!}; \quad 0 < \theta < 1.$$

6.103

$$f(x) = f(h) + f'(h)\cdot\frac{x-h}{1!} + f''(h)\cdot\frac{(x-h)^2}{2!} + \ldots + f^{(n)}(h)\frac{(x-h)^n}{n!} + R_n$$

$$R_n = f^{(n+1)}\{h + \theta(x-h)\}\frac{(x-h)^{n+1}}{(n+1)!};\quad 0 < \theta < 1.$$

6.104 Maclaurin's theorem:

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + \ldots + f^{(n)}(0)\frac{x^n}{n!} + R_n$$

$$R_n = f^{(n+1)}(\theta x)\frac{x^{n+1}}{(n+1)!}(1-\theta)^n;\quad 0 < \theta < 1.$$

6.105 Lagrange's theorem. Given:

$$y = z + x\phi(y).$$

The expansion of $f(y)$ in powers of x is:

$$f(y) = f(z) + x\phi(z)f'(z) + \frac{x^2}{2!}\frac{d}{dz}\left[\{\phi(z)\}^2 f'(z)\right]$$

$$+ \ldots + \frac{x^n}{n!}\frac{d^{n-1}}{dz^{n-1}}\left[\{\phi(z)\}^n f'(z)\right] + \ldots$$

SYMBOLIC REPRESENTATION OF INFINITE SERIES

6.150 The infinite series:

$$f(x) = 1 + a_1x + \frac{1}{2!}a_2x^2 + \frac{1}{3!}a_3x^3 + \ldots + \frac{1}{k!}a_kx^k + \ldots$$

may be written:

$$f(x) = e^{ax},$$

where a^k is interpreted as equivalent to a_k.

6.151 The infinite series, written without factorials,

$$f(x) = 1 + a_1x + a_2x^2 + \ldots + a_kx^k + \ldots$$

may be written:

$$f(x) = \frac{1}{1-ax},$$

where a^k is interpreted as equivalent to a_k.

6.152 Symbolic form of Taylor's theorem:

$$f(x+h) = e^{h\frac{d}{dx}}f(x).$$

6.153 Taylor's theorem for functions of many variables:

$$f(x_1+h_1, x_2+h_2, \ldots) = e^{h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \ldots} f(x_1, x_2, \ldots)$$

$$= f(x_1, x_2, \ldots) + h_1\frac{\partial f}{\partial x_1} + h_2\frac{\partial f}{\partial x_2} + \ldots$$

$$+ \frac{h_1^2}{2!}\frac{\partial^2 f}{\partial x_1^2} + \frac{2}{2!}h_1h_2\frac{\partial^2 f}{\partial x_1\partial x_2} + \frac{h_2^2}{2!}\frac{\partial^2 f}{\partial x_2^2} + \ldots$$

$$+ \ldots$$

TRANSFORMATION OF INFINITE SERIES

Series which converge slowly may often be transformed to more rapidly converging series by the following methods.

6.20 Euler's transformation formula:

$$S = a_0 + a_1 x + a_2 x^2 + \dots\dots\dots\dots$$

$$= \frac{1}{1-x} a_0 + \frac{1}{1-x} \sum_{k=1}^{\infty} \left(\frac{x}{1-x}\right)^k \Delta^k a_0,$$

where:

$$\Delta a_0 = a_1 - a_0,$$

$$\Delta^2 a_0 = \Delta a_1 - \Delta a_0 = a_2 - 2a_1 + a_0,$$

$$\Delta^3 a_0 = \Delta^2 a_1 - \Delta^2 a_0 = a_3 - 3a_2 + 3a_1 - a_0,$$

$$\dots\dots\dots\dots$$

$$\dots\dots\dots\dots$$

$$\Delta^k a_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{k+n-m}.$$

The second series may converge more rapidly than the first.

Example 1.

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{2k+1},$$

$$x = -1, \quad a_k = \frac{1}{2k+1},$$

$$S = \frac{1}{2} \sum_{k=0}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \dots (2k+1)}.$$

Example 2.

$$S = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1} = \log 2,$$

$$x = -1, \quad a_k = \frac{1}{k+1},$$

$$S = \sum_{k=1}^{\infty} \frac{1}{k2^k},$$

6.21 Markoff's transformation formula. (Differenzenrechnung, p. 180.)

$$\sum_{k=0}^{n} a_k x^k - \left(\frac{x}{1-x}\right)^m \sum_{k=0}^{n} x^k \Delta^m a_k = \sum_{k=0}^{m} \frac{x^k}{(1-x)^{k+1}} \Delta^k a_0 - \sum_{k=0}^{m} \frac{x^{k+n}}{(1-x)^{k+1}} \Delta^k a_n.$$

6.22 Kummer's transformation.

$A_0, A_1, A_2, \ldots$ is a sequence of positive numbers such that

$$\lambda_m = A_m - A_{m+1}\frac{a_{m+1}}{a_m},$$

and

$$\operatorname*{Limit}_{m\to\infty} \lambda_m,$$

approaches a definite positive value. Usually this limit can be taken as unity. If not, it is only necessary to divide A_m by this limit:

$$\alpha = \operatorname*{Limit}_{m\to\infty} A_m a_m.$$

Then:

$$\sum_{m=n}^{\infty} a_m = (A_n a_n - \alpha) + \sum_{m=n}^{\infty}(1 - \lambda_m)a_m.$$

Example 1.

$$S = \sum_{m=1}^{\infty}\frac{1}{m^2}$$

$$A_m = m, \quad \lambda_m = \frac{m}{m+1}, \quad \operatorname*{Limit}_{m\to\infty} \lambda_m = 1,$$

$$\alpha = 0$$

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = 1 + \sum_{m=1}^{\infty}\frac{1}{(m+1)m^2}.$$

Applying the transformation to the series on the right:

$$A_m = \frac{m}{2}, \quad \lambda_m = \frac{m}{m+2}, \quad \alpha = 0,$$

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = 1 + \frac{1}{2^2} + 2\sum_{m=1}^{\infty}\frac{1}{m^2(m+1)(m+2)}.$$

Applying the transformation n times:

$$\sum_{m=n+1}^{\infty}\frac{1}{m^2} = n!\sum_{m=1}^{\infty}\frac{1}{m^2(m+1)(m+2)\ldots(m+n)}.$$

Example 2.

$$S = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{2m-1},$$

$$A_m = \frac{1}{2}, \quad \lambda_m = \frac{2m}{2m+1}, \quad \alpha = 0,$$

$$S = \frac{1}{2} + \sum^{\infty}(-1)^{m-1}\frac{1}{4m^2-1}.$$

Applying the transformation again, with:

$$A_m = \frac{1}{2}\frac{2m+1}{2m-1}, \quad \lambda_m = \frac{4m^2+1}{4m^2-1}, \quad \alpha = 0,$$

$$S = 1 - 2\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(4m^2-1)^2}.$$

Applying the transformation again, with:

$$A_m = \frac{1}{2}\frac{2m+1}{2m-3}, \quad \lambda_m = \frac{4m^2+3}{4m^2-9}, \quad \alpha = 0,$$

$$S = \frac{4}{3} + 24\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(4m^2-1)^2(4m^2-9)}.$$

Example 3.

$$S = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(2m-1)^3},$$

$$A_m = \frac{2m-1}{2(2m-3)}, \quad \lambda_m = \frac{4m^2-4m+1}{(2m-3)(2m+1)}, \quad \alpha = 0,$$

$$S = \frac{5}{6} + 4\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{(2m-1)(2m+3)(2m+1)^3}.$$

6.23 Leclert's modification of Kummer's transformation. With the same notation as in **6.22** and,

$$\operatorname*{Limit}_{m\to\infty} \lambda_m = \omega,$$

$$\sum_{n=0}^{\infty} a_n = a_0 + \frac{A_1 a_1}{\lambda_1} - \frac{\alpha}{\omega} + \sum_{m=1}^{\infty}\left(\frac{1}{\lambda_{m+1}} - \frac{1}{\lambda_m}\right)A_{m+1}\,a_{m+1}.$$

Example 1.

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1},$$

$$a_0 = 0, \quad A_m = 1, \quad \omega = 2, \quad \alpha = 0, \quad \lambda_m = \frac{4m}{2m+1},$$

$$S = \frac{3}{4} + \frac{1}{4}\sum_{m=1}^{\infty}(-1)^{m-1}\frac{1}{m(2m+1)(m+1)}.$$

Applying the transformation to the series on the right, with:

$$a_0 = 0, \quad A_m = \frac{2m+1}{m+1}, \quad \lambda_m = \frac{(2m+1)^2}{(m+1)(m+2)}, \quad \omega = 4, \quad \alpha = 0,$$

$$S = \frac{19}{24} + \frac{9}{2}\sum_{m=1}^{\infty}(-1)^m \frac{1}{m(m+2)(2m+1)^2(2m+3)^2}.$$

6.26 Reversion of series. The power series:

$$z = x - b_1x^2 - b_2x^3 - b_3x^4 - \ldots\ldots$$

may be reversed, yielding:

$$x = z + c_1z^2 + c_2z^3 + c_3z^4 + \ldots\ldots$$

where:

$c_1 = b_1,$

$c_2 = b_2 + 2b_1^2,$

$c_3 = b_3 + 5b_1b_2 + 5b_1^3,$

$c_4 = b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4,$

$c_5 = b_5 + 7(b_1b_4 + b_2b_3) + 28(b_1^2b_3 + b_1b_2^2) + 84b_1^3b_2 + 42b_1^5,$

$$c_6 = b_6 + 4(2b_1b_5 + 2b_2b_4 + b_3^2) + 12(3b_1^2b_4 + 6b_1b_2b_3 + b_2^3) \\ + 60(2b_1^3b_3 + 3b_1^2b_2^2) + 330b_1^4b_2 + 132b_1^6,$$

$$c_7 = b_7 + 9(b_1b_6 + b_2b_5 + b_3b_4) + 45(b_1^2b_5 + b_1b_3^2 + b_2^2b_3 + 2b_1b_2b_4) \\ + 165(b_1^3b_4 + b_1b_2^3 + 3b_1^2b_2b_3) + 495(b_1^4b_3 + 2b_1^3b_2^2) \\ + 1287b_1^5b_2 + 429b_1^7$$

Van Orstrand (Phil. Mag. 19, 366, 1910) gives the coefficients of the reversed series up to c_{12}.

6.30 Binomial series.

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

$$+ \frac{n!}{(n-k)!k!}x^k + \ldots = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots \binom{n}{k}x^k + \ldots$$

6.31 Convergence of the binomial series.

The series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. When $x = 1$, the series converges for $n > -1$ and diverges for $n \leqslant -1$. It is absolutely convergent only for $n > 0$.

When $x = -1$ it is absolutely convergent for $n > 0$, and divergent for $n < 0$.

6.32 Special cases of the binomial series.

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = b^n\left(1+\frac{a}{b}\right)^n.$$

If $\left|\frac{b}{a}\right| < 1$ put $x = \frac{b}{a}$ in **6.30**; if $\left|\frac{b}{a}\right| > 1$ put $x = \frac{a}{b}$ in **6.30**.

6.33

1. $(1+x)^{\frac{n}{m}} = 1 + \frac{n}{m}x - \frac{n(m-n)}{2!m^2}x^2 + \frac{n(m-n)(2m-n)}{3!m^3}x^3 - \dots \dots + (-1)^k \frac{n(m-n)(2m-n)\dots\dots[(k-1)m-n]}{k!m^k}x^k + \dots\dots$

2. $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots\dots$
3. $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots\dots$
4. $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1\cdot 1}{2\cdot 4}x^2 + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}x^3 - \frac{1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 + \dots$
5. $\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 - \dots$
6. $(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1\cdot 2}{3\cdot 6}x^2 + \frac{1\cdot 2\cdot 5}{3\cdot 6\cdot 9}x^3 - \frac{1\cdot 2\cdot 5\cdot 8}{3\cdot 6\cdot 9\cdot 12}x^4 + \dots\dots$
7. $(1+x)^{-\frac{1}{3}} = 1 - \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 - \frac{1\cdot 4\cdot 7}{3\cdot 6\cdot 9}x^3 + \frac{1\cdot 4\cdot 7\cdot 10}{3\cdot 6\cdot 9\cdot 12}x^4 - \dots$
8. $(1+x)^{\frac{3}{2}} = 1 + \frac{3}{2}x + \frac{3\cdot 1}{2\cdot 4}x^2 - \frac{3\cdot 1\cdot 1}{2\cdot 4\cdot 6}x^3 + \frac{3\cdot 1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 8}x^4 - \frac{3\cdot 1\cdot 1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^5 + \dots\dots$
9. $(1+x)^{-\frac{3}{2}} = 1 - \frac{3}{2}x + \frac{3\cdot 5}{2\cdot 4}x^2 - \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}x^3 + \dots\dots$
10. $(1+x)^{\frac{1}{4}} = 1 + \frac{1}{4}x - \frac{3}{32}x^2 + \frac{7}{128}x^3 - \frac{77}{2048}x^4 + \dots$
11. $(1+x)^{-\frac{1}{4}} = 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \frac{105}{2048}x^4 - \dots$
12. $(1+x)^{\frac{1}{5}} = 1 + \frac{1}{5}x - \frac{2}{25}x^2 + \frac{6}{125}x^3 - \frac{21}{625}x^4 + \dots\dots$

13. $(1+x)^{-\frac{1}{5}} = 1 - \frac{1}{5}x + \frac{3}{25}x^2 - \frac{11}{125}x^3 + \frac{44}{625}x^4 - \ldots$

14. $(1+x)^{\frac{1}{6}} = 1 + \frac{1}{6}x - \frac{5}{72}x^2 + \frac{55}{1296}x^3 - \frac{935}{31104}x^4 + \ldots$

15. $(1+x)^{-\frac{1}{6}} = 1 - \frac{1}{6}x + \frac{7}{72}x^2 - \frac{91}{1296}x^3 + \frac{1729}{31104}x^4 - \ldots$

6.350

1. $\frac{x}{1-x} = \frac{x}{1+x} + \frac{2x^2}{1+x^2} + \frac{4x^4}{1+x^4} + \frac{8x^8}{1+x^8} + \ldots \qquad [x^2 < 1].$

2. $\frac{x}{1-x} = \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \frac{x^4}{1-x^8} + \ldots \qquad [x^2 < 1].$

3. $\frac{1}{x-1} = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \ldots \qquad [x^2 > 1].$

6.351

1. $$\left\{1 + \sqrt{1+x}\right\}^n = 2^n\left\{1 + n\left(\frac{x}{4}\right) + \frac{n(n-3)}{2!}\left(\frac{x}{4}\right)^2 + \frac{n(n-4)(n-5)}{3!}\left(\frac{x}{4}\right)^3 + \ldots\right\}. \qquad [x^2 < 1].$$

n may be any real number.

2. $$\left(x + \sqrt{1+x^2}\right)^n = 1 + \frac{n^2}{2!}x^2 + \frac{n^2(n^2-2^2)}{4!}x^4 + \frac{n^2(n^2-2^2)(n^2-4^2)}{6!}x^6 + \ldots + \frac{n}{1!}x + \frac{n(n^2-1^2)}{3!}x^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!}x^5 + \ldots \qquad [x^2 < 1].$$

6.352 If a is a positive integer:

$$\frac{1}{a} + \frac{1}{a(a+1)}x + \frac{1}{a(a+1)(a+2)}x^2 + \ldots = \frac{(a-1)!}{x^a}\left\{e^x - \sum_{n=0}^{a-1}\frac{x^n}{n!}\right\}.$$

6.353 If a and b are positive integers, and $a<b$:

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}x^2 + \ldots$$

$$= (b-a)\binom{b-1}{a-1}\left\{\frac{(-1)^{b-a}\log(1-x)}{x^b}(1-x)^{b-a-1} + \frac{1}{x^a}\sum^{b-a}(-1)^k\binom{b-a-1}{k-1}\sum^{a+k-1}\frac{x^{n-k}}{n}\right\}.$$

POLYNOMIAL SERIES

6.360
$$\frac{b_0 + b_1x + b_2x^2 + b_3x^3 + \dots}{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots} = \frac{1}{a_0}(c_0 + c_1x + c_2x^2 + \dots),$$

$$c_0 - b_0 = 0,$$

$$c_1 + \frac{c_0a_1}{a_0} - b_1 = 0,$$

$$c_2 + \frac{c_1a_1}{a_0} + \frac{c_0a_2}{a_0} - b_2 = 0,$$

$$c_3 + \frac{c_2a_1}{a_0} + \frac{c_1a_2}{a_0} + \frac{c_0a_3}{a_0} - b_3 = 0.$$

. . . . . .

. . . . . .

$$c_n = \frac{(-1)^n}{a_0^n}\begin{vmatrix} (a_1b_0 - a_0b_1) & a_0 & 0 & \dots\dots & 0 \\ (a_2b_0 - a_0b_2) & a_1 & a_0 & \dots\dots & 0 \\ (a_3b_0 - a_0b_3) & a_2 & a_1 & \dots\dots & 0 \\ \dots\dots\dots & & & & \\ \dots\dots\dots & & & & \\ (a_{n-1}b_0 - a_0b_{n-1}) & a_{n-2} & a_{n-3} & \dots\dots & a_0 \\ (a_nb_0 - a_0b_n) & a_{n-1} & a_{n-2} & \dots\dots & a_1 \end{vmatrix}$$

6.361
$$(a_0 + a_1x + a_2x^2 + \dots)^n = c_0 + c_1x + c_2x^2 + \dots$$

$c_0 = a_0^n,$

$a_0c_1 = na_1c_0,$

$2a_0c_2 = (n-1)a_1c_1 + 2na_2c_0,$

$3a_0c_3 = (n-2)a_1c_2 + (2n-1)a_2c_1 + 3na_3c_0.$

. . . . . .

. . . . . . *cf.* 6.37.

6.362
$$y = a_1x + a_2x^2 + a_3x^3 + \dots$$

$$b_1y + b_2y^2 + b_3y^3 + \dots = c_1x + c_2x^2 + c_3x^3 + \dots$$

$c_1 = a_1b_1,$

$c_2 = a_2b_1 + a_1^2b_2,$

$c_3 = a_3b_1 + 2a_1a_2b_2 + a_1^3b_3,$

$c_4 = a_4b_1 + a_2^2b_2 + 2a_1a_3b_2 + 3a_1^2a_2b_3 + a_1^4b_4.$

. . . .

. . . .

6.363
$$e^{a_1x + a_2x^2 + a_3x^3 + \dots} = 1 + c_1x + c_2x^2 + \dots$$

$c_1 = a_1,$

$c_2 = a_2 + \frac{1}{2}a_1^2,$

$$c_3 = a_3 + a_1a_2 + \frac{1}{6}a_1^3,$$

$$c_4 = a_4 + a_1a_3 + \frac{1}{2}a_2^2 + \frac{1}{2}a_2a_1^2 + \frac{1}{24}a_1^4.$$

$$\cdots\cdots$$

$$\cdots\cdots$$

6.364

$$\log(1 + a_1x + a_2x^2 + a_3x^3 + \ldots) = c_1x + c_2x^2 + c_3x^3 + \ldots,$$

$$a_1 = c_1,$$

$$2a_2 = a_1c_1 + 2c_2,$$

$$3a_3 = a_2c_1 + 2a_1c_2 + 3c_3,$$

$$4a_4 = a_3c_1 + 2a_2c_2 + 3a_1c_3 + 4c_4.$$

$$\cdots$$

$$c_1 = a_1,$$

$$c_2 = a_2 - \frac{1}{2}c_1a_1,$$

$$c_3 = a_3 - \frac{1}{3}c_1a_2 - \frac{2}{3}c_2a_1,$$

$$c_4 = a_4 - \frac{1}{4}c_1a_3 - \frac{2}{4}c_2a_2 - \frac{3}{4}c_3a_1.$$

$$\cdots$$

6.365

$$y = a_1x + a_2x^2 + a_3x^3 + \ldots$$

$$z = b_1x + b_2x^2 + b_3x^3 + \ldots$$

$$yz = c_2x^2 + c_3x^3 + c_4x^4 + \ldots$$

$$c_2 = a_1b_1,$$

$$c_3 = a_1b_2 + a_2b_1,$$

$$c_4 = a_1b_3 + a_2b_2 + a_3b_1.$$

$$\cdots$$

$$c_k = a_1b_{k-1} + a_2b_{k-2} + a_3b_{k-3} + \ldots a_{k-1}b_1.$$

6.37. The Multinomial Theorem.

The general term in the expansion of

(1) $$(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots)^n$$

where n is positive or negative, integral or fractional, is,

(2) $$\frac{n(n-1)(n-2)\ldots(p+1)}{c_1!\,c_2!\,c_3!\ldots}\,a_0^p a_1^{c_1} a_2^{c_2} a_3^{c_3}\ldots x^{c_1+2c_2+3c_3+\ldots}\ldots$$

where

$$p + c_1 + c_2 + c_3 + \ldots\ldots = n.$$

$c_1, c_2, c_3, \ldots\ldots$ are positive integers.

If n is a positive integer, and hence p also, the general term in the expansion

(3) $$\frac{n!}{p!\,c_1!\,c_2!\,\ldots}\,a_0^{p}a_1^{c_1}a_2^{c_2}a_3^{c_3}\,\ldots\,x^{c_1+2c_2+3c_3+\ldots}\ldots$$

The coefficient of x^k (k an integer) in the expansion of (1) is found by taking the sum of all the terms (2) or (3) for the different combinations of p, $c_1, c_2, c_3, \ldots$ which satisfy

$$c_1 + 2c_2 + 3c_3 + \ldots = k,$$
$$p + c_1 + c_2 + c_3 + \ldots = n.$$

cf. 6.361.

In the following series the coefficients B_n are Bernoulli's numbers (6.902) and the coefficients E_n, Euler's numbers (6.903).

6.400

1. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad [x^2 < \infty].$$

2. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad [x^2 < \infty].$$

3. $$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \ldots$$
$$= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n x^{2n-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \frac{1}{4725}x^7 - \ldots$$
$$= \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}B_n}{(2n)!} x^{2n-1} \qquad [x^2 < \pi^2].$$

5. $$\sec x = 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4 + \frac{61}{6!}x^6 + \ldots = \sum_{n=0}^{\infty} \frac{E_n}{(2n)!} x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

6. $$\csc x = \frac{1}{x} + \frac{1}{3!}x + \frac{7}{3\cdot 5!}x^3 + \frac{31}{3\cdot 7!}x^5 + \ldots$$
$$= \frac{1}{x} + \sum_{n=0}^{\infty} \frac{2(2^{2n+1}-1)}{(2n+2)!} B_{n+1} x^{2n+1} \qquad [x^2 < \pi^2].$$

6.41

1. $$\sin^{-1} x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7 + \ldots \qquad [x^2 \leq 1].$$
$$= \frac{\pi}{2} - \cos^{-1} x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1},$$

2. $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$ (Gregory's Series) $\left[x^2 \leqslant 1\right]$

$$= \frac{\pi}{2} - \cot^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

3. $\tan^{-1} x = \frac{x}{1+x^2}\left\{1 + \frac{2}{3}\frac{x^2}{1+x^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{x^2}{1+x^2}\right)^2 + \ldots\right\}$

$$= \frac{x}{1+x^2}\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}\left(\frac{x^2}{1+x^2}\right)^n \qquad x^2 < \infty.$$

4. $\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \ldots$

$$= \frac{\pi}{2} - \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)x^{2n+1}} \qquad \left[x^2 \geqslant 1\right].$$

5. $\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{2\cdot 3}\frac{1}{x^3} - \frac{1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\frac{1}{x^7} - \ldots$

$$= \frac{\pi}{2} - \csc^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{-2n-1} \qquad \left[x > 1\right].$$

6.42

1. $(\sin^{-1} x)^2 = x^2 + \frac{2}{3}\frac{x^4}{2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^6}{3} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\frac{x^8}{4} + \ldots$

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n+1)!\,(n+1)} x^{2n+2} \qquad \left[x^2 \leqslant 1\right].$$

2. $(\sin^{-1} x)^3 = x^3 + \frac{3!}{5!}3^2\left(1 + \frac{1}{3^2}\right)x^5 + \frac{3!}{7!}3^2 5^2\left(1 + \frac{1}{3^2} + \frac{1}{5^2}\right)x^7 + \ldots \qquad \left[x^2 \leqslant 1\right].$

3. $(\tan^{-1} x)^p = p!\sum_{k_0=1}^{\infty}(-1)^{k_0-1}\frac{x^{2k_0+p-2}}{2k_0+p-2}\prod_{a=1}^{p-1}\left(\sum_{k_a=1}^{k_{a-1}}\frac{1}{2k_a+p-a-2}\right).$

(Schwatt, Phil. Mag. 31, p. 490, 1916).

4. $\sqrt{1-x^2}\sin^{-1} x = x - \frac{x^3}{3} + \frac{2}{3\cdot 5}x^5 - \frac{2\cdot 4}{3\cdot 5\cdot 7}x^7 + \ldots$

$$= x + \sum_{n=1}^{\infty}(-1)^n \frac{2^{2n-2}[(n-1)!]^2}{(2n-1)!\,(2n+1)} x^{2n+1} \qquad \left[x^2 < 1\right].$$

5. $\frac{\sin^{-1} x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots$

$$= \sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2}{(2n+1)!}x^{2n+1} \qquad \left[x^2 < 1\right].$$

0.43

1. $\log \sin x = \log x - \left\{ \frac{1}{6}x^2 + \frac{1}{180}x^4 + \frac{1}{2835}x^6 + \dots \right\}$

$$= \log x - \sum_{n=1}^{\infty} \frac{2^{2n-1}}{n(2n)!} B_n x^{2n} \qquad \left[x^2 < \pi^2\right].$$

2. $\log \cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{17}{2520}x^8 - \dots$

$$= -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)B_n}{n(2n)!} x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

3. $\log \tan x = \log x + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots$

$$= \log x + \sum_{n=1}^{\infty} \frac{(2^{2n-1}-1)2^{2n}}{n(2n)!} B_n x^{2n} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $\log \cos x = -\frac{1}{2}\left\{ \sin^2 x + \frac{1}{2}\sin^4 x + \frac{1}{3}\sin^6 x + \dots \right\}$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} \sin^{2n} x \qquad \left[x^2 < \frac{\pi^2}{4}\right]$$

0.44

1. $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \qquad \left[-1 < x \leq 1\right].$$

$[\log(1+x)]^n$ see **7.300.**

2. $\log(x + \sqrt{1+x^2}) = x - \frac{1 \cdot 1}{2 \cdot 3}x^3 + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 5}x^5 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7}x^7 + \dots$

$$= x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{2^{2n-1} n!\,(n-1)!} \frac{x^{2n+1}}{(2n+1)} \qquad \left[-1 \leq x \leq 1\right].$$

3. $\log(1 + \sqrt{1+x^2}) = \log 2 + \frac{1 \cdot 1}{2 \cdot 2}x^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 4}x^4 + \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 6}x^6 - \dots$

$$= \log 2 - \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{2^{2n-1} n!(n-1)!} \frac{x^{2n}}{2n} \qquad \left[x^2 \leq 1\right].$$

4. $\log(1+\sqrt{1+x^2}) = \log x + \frac{1}{x} - \frac{1\cdot 1}{2\cdot 3}\frac{1}{x^3} + \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 5}\frac{1}{x^5} - \ldots$

$$= \log x + \frac{1}{x} + \sum_{n=1}^{\infty}(-1)^n \frac{(2n-1)!}{2^{2n-1}n!\,(n-1)!}\,\frac{x^{-2n-1}}{(2n+1)} \qquad \left[x^2 \geq 1\right].$$

5. $\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \ldots$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-1)^n}{n} \qquad \left[0 < x \leq 2\right].$$

6. $\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \ldots$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{x-1}{x}\right)^n \qquad \left[x > \tfrac{1}{2}\right].$$

7. $\log x = 2\left\{\frac{x-1}{x+1} + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \ldots\right\}$

$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1} \qquad \left[x > 0\right].$$

8. $\log\frac{1+x}{1-x} = 2\left\{x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots\right\}$

$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}x^{2n+1} \qquad \left[x^2 < 1\right].$$

9. $\log\frac{x+1}{x-1} = 2\left\{\frac{1}{x} + \frac{1}{3}\frac{1}{x^3} + \frac{1}{5}\frac{1}{x^5} + \ldots\right\}$

$$= 2\sum_{n=0}^{\infty}\frac{1}{(2n+1)x^{2n+1}} \qquad \left[x^2 > 1\right].$$

10. $\sqrt{1+x^2}\log(x+\sqrt{1+x^2}) = x + \frac{1}{3}x^3 - \frac{1\cdot 2}{3\cdot 5}x^5 + \frac{1\cdot 2\cdot 4}{3\cdot 5\cdot 7}x^7 - \ldots$

$$= x - \sum_{n=1}^{\infty}(-1)^n\frac{(n-1)!\,2^{2n-1}n!}{(2n+1)!}x^{2n+1} \qquad \left[x^2 \leq 1\right].$$

11. $\frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} = x - \frac{2}{3}x^3 + \frac{2\cdot 4}{3\cdot 5}x^5 - \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}(n!)^2}{(2n+1)!}x^{2n+1} \qquad \left[x^2 < 1\right].$$

12. $\left\{\log(x+\sqrt{1+x^2})\right\}^2 = \frac{x^2}{1} - \frac{2}{3}\frac{x^4}{2} + \frac{2\cdot 4}{3\cdot 5}\frac{x^6}{3} - \ldots$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n-2}(n-1)!\,(n-1)!}{(2n-1)!}\frac{x^{2n}}{n} \qquad \left[x^2 < 1\right]$$

13. $\frac{1}{2}\left\{\log(1+x)\right\}^2 = \frac{1}{2}s_1x^2 - \frac{1}{3}s_2x^3 + \frac{1}{4}s_3x^4 - \ldots \qquad [x^2<1].$

where $s_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$ (See 1.876).

14. $\frac{1}{6}\left\{\log(1+x)\right\}^3 = \frac{1}{3}\cdot\frac{1}{2}s_1x^3 - \frac{1}{4}\left(\frac{1}{2}s_1+\frac{1}{3}s_2\right)x^4$

$$+\frac{1}{5}\left(\frac{1}{2}s_1+\frac{1}{3}s_2+\frac{1}{4}s_3\right)x^5 - \ldots \qquad [x^2<1].$$

15. $\frac{\log(1+x)}{(1+x)^n} = x - n(n+1)\left(\frac{1}{n}+\frac{1}{n+1}\right)\frac{x^2}{2!}$

$$+n(n+1)(n+2)\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}\right)\frac{x^3}{3!} - \ldots \qquad [x^2<1].$$

6.445 (See 6.705.)

1. $\frac{3}{4x} - \frac{1}{2x^2} + \frac{(1-x)^2}{2x^3}\log\frac{1}{1-x} = \frac{1}{1\cdot2\cdot3}+\frac{x}{2\cdot3\cdot4}+\frac{x^2}{3\cdot4\cdot5}+\ldots \qquad [x^2<1].$

2. $\frac{1}{4x}\left\{\frac{1+x}{\sqrt{x}}\log\frac{1+\sqrt{x}}{1-\sqrt{x}} + 2\log(1-x) - 2\right\} = \frac{1}{1\cdot2\cdot3}+\frac{x}{3\cdot4\cdot5}$

$$+\frac{x^2}{5\cdot6\cdot7}+\ldots \qquad [0<x<1].$$

3. $\frac{1}{2x}\left\{1 - \log(1+x) - \frac{1-x}{\sqrt{x}}\tan^{-1}x\right\} = \frac{1}{1\cdot2\cdot3}-\frac{x}{3\cdot4\cdot5}$

$$+\frac{x^2}{5\cdot6\cdot7}-\ldots \qquad [0<x\leqq1].$$

6.455

1. $-\log(1+x)\cdot\log(1-x) = x^2 + \left(1-\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{2}$

$$+\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}\right)\frac{x^6}{3}+\ldots \qquad [x^2<1].$$

2. $\frac{1}{2}\tan^{-1}x\cdot\log\frac{1+x}{1-x} = x^2 + \left(1-\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{3} + \left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}\right)\frac{x^{10}}{5}$

$$+\ldots \qquad [x^2<1].$$

3. $\frac{1}{2}\tan^{-1}x\cdot\log(1+x^2) = \left(1+\frac{1}{2}\right)\frac{x^3}{3} - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^5}{5}+\ldots \qquad [x^2<1].$

6.456

1. $\cos\left\{k\log(x+\sqrt{1+x^2})\right\} = 1 - \frac{k^2}{2!}x^2 + \frac{k^2(k^2+2^2)}{4!}x^4$

$$-\frac{k^2(k^2+2^2)(k^2+4^2)}{6!}x^6+\ldots \qquad x^2<1.$$

2. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ $\left[x^2 < \infty\right]$.

3. $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \ldots$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_n x^{2n-1} \qquad \left[x^2 < \frac{\pi^2}{4}\right].$$

4. $x \coth x = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{2}{945}x^6 - \ldots$

$$= 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n} B_n}{(2n)!} x^{2n} \qquad \left[x^2 < \pi^2\right].$$

5. $\operatorname{sech} x = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \ldots = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{E_n}{(2n)!} x^{2n}$ $\left[x^2 < \frac{\pi^2}{4}\right]$.

6. $x \operatorname{csch} x = 1 - \frac{1}{6}x^2 + \frac{7}{360}x^4 - \frac{31}{15120}x^6 + \ldots$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{2(2^{2n-1}-1)}{(2n)!} B_n x^{2n} \qquad \left[x^2 < \pi^2\right].$$

0.475

1. $\cosh x \cos x = 1 - \frac{2^2}{4!}x^4 + \frac{2^4}{8!}x^8 - \frac{2^6}{12!}x^{12} + \ldots$

2. $\sinh x \sin x = \frac{2^2}{2!}x^2 - \frac{2^4}{6!}x^6 + \frac{2^6}{10!}x^{10} - \ldots\ldots$

0.476

1. $e^{x\cos\theta} \cos(x \sin\theta) = \sum_{n=0}^{\infty} \frac{x^n \cos n\theta}{n!}$ $\left[x^2 < 1\right]$.

2. $e^{x\cos\theta} \sin(x \sin\theta) = \sum_{n=1}^{\infty} \frac{x^n \sin n\theta}{n!}$ $\left[x^2 < 1\right]$.

3. $\cosh(x\cos\theta) \cdot \cos(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n} \cos 2n\theta}{(2n)!}$ $\left[x^2 < 1\right]$.

4. $\sinh(x\cos\theta) \cdot \cos(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n+1} \cos(2n+1)\theta}{(2n+1)!}$ $\left[x^2 < 1\right]$.

5. $\cosh(x\cos\theta) \cdot \sin(x\sin\theta) = \sum_{n=0}^{\infty} \frac{x^{2n+1} \sin(2n+1)\theta}{(2n+1)!}$ $\left[x^2 < 1\right]$.

6. $\sinh(x\cos\theta) \cdot \sin(x\sin\theta) = \sum \frac{x^{2n} \sin 2n\theta}{(2n)!}$ $\left[x^2 < 1\right]$.

6.480

1. $\sinh^{-1} x = x - \frac{1}{2\cdot 3} x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5} x^5 - \dots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+1)} x^{2n+1} \qquad [x^2 < 1].$$

2. $\sinh^{-1} x = \log 2x + \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} + \dots$

$$= \log 2x + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{2^{2n}(n!)^2 2n} x^{-2n} \qquad [x^2 > 1].$$

3. $\cosh^{-1} x = \log 2x - \frac{1}{2}\frac{1}{2x^2} - \frac{1\cdot 3}{2\cdot 4}\frac{1}{4x^4} - \dots$

$$= \log 2x - \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n} x^{-2n} \qquad [x^2 > 1].$$

4. $\tanh^{-1} x = x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \frac{1}{7} x^7 + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \qquad [x^2 < 1].$

5. $\sinh^{-1} \frac{1}{x} = \frac{1}{x} - \frac{1}{2}\frac{1}{3x^3} + \frac{1\cdot 3}{2\cdot 4}\frac{1}{5x^5} - \dots$

$$= \operatorname{csch}^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{2^{2n}(n!)^2 (2n+1)} x^{-2n-1} \qquad [x^2 > 1].$$

6. $\cosh^{-1} \frac{1}{x} = \log \frac{2}{x} - \frac{1}{2}\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\frac{x^4}{4} - \dots$

$$= \operatorname{sech}^{-1} x = \log \frac{2}{x} - \sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2 2n} x^{2n} \qquad [x^2 < 1].$$

7. $\sinh^{-1} \frac{1}{x} = \log \frac{2}{x} + \frac{1}{2}\frac{x^2}{2} - \frac{1\cdot 3}{2\cdot 4}\frac{x^4}{4} + \dots$

$$= \operatorname{csch}^{-1} x = \log \frac{2}{x} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n)!}{2^{2n}(n!)^2 2n} x^{2n} \qquad [x^2 < 1].$$

8. $\tanh^{-1} \frac{1}{x} = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \dots$

$$= \coth^{-1} x = \sum_{n=0}^{\infty} \frac{x^{-2n-1}}{2n+1} \qquad [x^2 > 1].$$

6.490

1. $$\frac{1}{2\sinh x} = \sum_{n=0}^{\infty} e^{-x(2n+1)}.$$

2. $$\frac{1}{2\cosh x} = \sum_{n=0}^{\infty} (-1)^n e^{-x(2n+1)}.$$

3. $$\frac{1}{2}(\tanh x - 1) = \sum_{n=1}^{\infty} (-1)^n e^{-2nx}.$$

4. $$-\frac{1}{2}\log\tanh\frac{x}{2} = \sum_{n=0}^{\infty} \frac{1}{2n+1} e^{-x(2n+1)}.$$

6.491

$$\frac{1}{2} + \sum_{n=1}^{\infty} e^{-(nx)^2} = \frac{\sqrt{\pi}}{x}\left\{\frac{1}{2} + \sum_{n=1}^{\infty} e^{-\left(\frac{n\pi}{x}\right)^2}\right\}.$$

By means of this formula a slowly converging series may be transformed into a rapidly converging series.

6.495

1. $$\tan x = 2x\left\{\frac{1}{\left(\frac{\pi}{2}\right)^2 - x^2} + \frac{1}{\left(\frac{3\pi}{2}\right)^2 - x^2} + \frac{1}{\left(\frac{5\pi}{2}\right)^2 - x^2} + \cdots\cdots\right\}$$
$$= \sum_{n=1}^{\infty} \frac{8x}{(2n-1)^2\pi^2 - 4x^2}.$$

2. $$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} - \frac{2x}{(3\pi)^2 - x^2} - \cdots\cdots = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2x}{n^2\pi^2 - x^2}.$$

3. $$\sec x = \frac{\pi}{\left(\frac{\pi}{2}\right)^2 - x^2} - \frac{3\pi}{\left(\frac{3\pi}{2}\right)^2 - x^2} + \frac{5\pi}{\left(\frac{5\pi}{2}\right)^2 - x^2} - \cdots\cdots$$
$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4(2n-1)\pi}{(2n-1)^2\pi^2 - 4x^2}.$$

4. $$\csc x = \frac{1}{x} + \frac{2x}{\pi^2 - x^2} - \frac{2x}{(2\pi)^2 - x^2} + \frac{2x}{(3\pi)^2 - x^2} - \cdots\cdots$$
$$= \frac{1}{x} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2x}{n^2\pi^2 - x^2}.$$

By replacing x by ix the corresponding series for the hyperbolic functions may be written.

may be transformed into the infinite product

$$(1 + v_1)(1 + v_2)(1 + v_3) \ldots \ldots$$
$$= \prod_{n=1}^{\infty} (1 + v_n),$$

where

$$v_n = \frac{u_n}{1 + u_1 + u_2 + \ldots \ldots + u_{n-1}}.$$

6.600 The Gamma Function:

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^z}{1 + \frac{z}{n}},$$

z may have any real or complex value, except $0, -1, -2, -3, \ldots \ldots$

6.601

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

6.602

$$\gamma = \underset{m \to \infty}{\text{Limit}} \left\{ 1 + \frac{1}{2} + \frac{1}{3} + \ldots \ldots + \frac{1}{m} - \log m \right\}$$
$$= \int_0^{\infty} \left\{ \frac{e^{-t}}{1 - e^{-t}} - \frac{e^{-t}}{t} \right\} dt = 0.5772157 \ldots$$

6.603

$$\Gamma(z + 1) = z\Gamma(z),$$
$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}.$$

6.604 For z real and positive $= x$:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt,$$

$$\log \Gamma(1 + x) = \left(x + \frac{1}{2}\right) \log x - x + \frac{1}{2} \log 2\pi + \int_0^{\infty} \left\{ \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right\} e^{-xt} \frac{dt}{t}.$$

6.605 If $z = n$, a positive integer:

$$\Gamma(n) = (n - 1)!,$$

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \ldots \ldots (2n - 1)}{2^n} \sqrt{\pi},$$

6.606 The Beta Function. If x and y are real and positive:

$$B(x, y) = B(y, x) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,$$

$$B(x+1, y) = \frac{x}{x+y}B(x, y),$$

$$B(x, 1-x) = \frac{\pi}{\sin \pi x}.$$

6.610 For x real and positive:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \sum_{n=0}^{\infty}\left(\frac{1}{x+n} - \frac{1}{n+1}\right).$$

6.611

$$\psi(x+1) = \frac{1}{x} + \psi(x),$$

6.612

$$\psi(1-x) = \psi(x) + \pi \cot \pi x.$$

$$\psi(\tfrac{1}{2}) = -\gamma - 2\log 2,$$
$$\psi(1) = -\gamma,$$
$$\psi(2) = 1 - \gamma,$$
$$\psi(3) = 1 + \frac{1}{2} - \gamma,$$
$$\psi(4) = 1 + \frac{1}{2} + \frac{1}{3} - \gamma.$$

. . . . . .

. . . . . .

6.613

$$\psi(x) = \int_0^{\infty}\left\{\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right\}dt$$
$$= -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t}\,dt.$$

6.620

$$\beta(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}.$$

6.621

$$\beta(x+1) + \beta(x) = \frac{1}{x},$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}.$$

6.622

$$\beta(1) = \log 2,$$

$$\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}.$$

6.630 Gauss's Π Function:

1. $\Pi(k, z) = k^z \prod_{n=1}^{k} \frac{n}{z+n}.$
2. $\Pi(k, z+1) = \Pi(k, z) \cdot \dfrac{1+z}{1+\dfrac{1+z}{k}}.$
3. $\Pi(z) = \lim_{k \to \infty} \Pi(k, z).$
4. $\Pi(z) = \Gamma(z+1).$
5. $\Pi(-z)\,\Pi(z-1) = \pi \csc \pi z.$
6. $\Pi\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$

6.631 If z is an integer, n,

$$\Pi(n) = n!$$

DEFINITE INTEGRALS EXPRESSED AS INFINITE SER[IES]

6.700

$$\int_0^x e^{-x^2}\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} x^{2k+1}.$$

$$= e^{-x^2} \sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1\cdot 3\cdot 5 \cdot \ldots \cdot (2k+1)}$$

Darling (Quarterly Journal, 49, p. 36, 1920) has obtained an approximation to this integral:

$$\frac{\sqrt{\pi}}{2} - \frac{2}{\sqrt{\pi}} \tan^{-1} \left\{ e^{\sqrt{\pi}}(1 + x^2 e^{-\sqrt{\pi}})^2 \right\}^{-x}$$

Fresnel's Integrals:

6.701 $$\int_0^x \cos(x^2)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!\,(4k+1)} x^{4k+1}$$

$$= \cos(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} x^{4k+1}}{1\cdot 3\cdot 5\ \ldots\ (4k+1)}$$

$$+ \sin(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{4k+3}}{1\cdot 3\cdot 5\ \ldots\ (4k+3)}.$$

6.702 $$\int_0^x \sin(x^2)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!\,(4k+3)} x^{4k+3}$$

$$= \sin(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k}}{1\cdot 3\cdot 5\ \ldots\ (4k+1)} x^{4k+1}$$

$$- \cos(x^2) \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1} x^{4k+3}}{1\cdot 3\cdot 5\ \ldots\ (4k+3)}.$$

6.703 $$\int_0^1 \frac{t^{a-1}}{1+t^b}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{1}{a+nb}$$

6.704 $$\frac{1}{(k-1)!}\int_0^1 \frac{t^{a-1}(1-t)^{k-1}}{1-xt^b}\,dt$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(a+nb)\,(a+nb+1)\,(a+nb+2)\ \ldots\ (a+nb+k-1)}$$

(Special cases, **6.445** and **6.022**). $[b>0,\ x^2 \leq 1]$.

6.705 $$\int_0^x e^{-t}\,t^{y-1}\,dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+y}}{n!(n+y)} = e^{-x}\sum_{n=0}^{\infty} \frac{x^{n+y}}{y(y+1)\ \ldots\ (y+n)}.$$

6.706 If the sum of the series,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad [0 < x < 1]$$

is known, then

$$\sum_{n=0}^{\infty} \frac{c_n x^n}{(a+nb)\,(a+nb+1)\,(a+nb+2)\ \ldots\ \ldots\ (a+nb+k-1)} \qquad [b>0]$$

$$= \frac{1}{(k-1)!}\int_0^1 t^{a-1}(1-t)^{k-1}$$

6.707 $$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \cdot dx = \frac{1}{2} \int_0^{2\pi} (\pi - t) \sum_{n=0}^{\infty} f(t + 2n\pi) \cdot dt.$$

Example 1. $f(x) = e^{-kx}$ [$k>0$].

1. $$\frac{1}{k} + 2k \sum_{n=1}^{\infty} \frac{1}{k^2 + n^2} = \pi \frac{e^{k\pi} + e^{-k\pi}}{e^{k\pi} - e^{-k\pi}}.$$

Replacing k by $\frac{k}{2}$, and subtracting,

2 $$\frac{1}{k} + 2k \sum_{n=1}^{\infty} (-1)^n \frac{1}{k^2 + n^2} = \frac{2\pi}{e^{k\pi} - e^{-k\pi}}.$$

Example 2. With $f(x) = e^{-\lambda x} \cos \mu x$ and $e^{-\lambda x} \sin \mu x$.

3. $$\frac{\lambda}{\lambda^2 + \mu^2} + \sum_{n=1}^{\infty} \left\{ \frac{\lambda}{\lambda^2 + (n - \mu)^2} + \frac{\lambda}{\lambda^2 + (n + \mu)^2} \right\} = \frac{\pi \sinh 2\lambda\pi}{\cosh 2\lambda\pi - \cos 2\mu\pi}.$$

4. $$\frac{\mu}{\lambda^2 + \mu^2} - \sum_{n=1}^{\infty} \left\{ \frac{n - \mu}{\lambda^2 + (n - \mu)^2} + \frac{n + \mu}{\lambda^2 + (n + \mu)^2} \right\} = \frac{\pi \sin 2\mu\pi}{\cosh 2\lambda\pi - \cos 2\mu\pi}.$$

6.709 If the sum of the series,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

is known, then

$$a_0 + a_1 y + a_2 y(y + 1) + a_3 y(y + 1)(y + 2) + \ldots\ldots = \frac{\int_0^{\infty} e^{-t} t^{y-1} f(t)\, dt}{\Gamma(y)}.$$

6.710 The complete elliptic integral of the first kind:

$$K = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}$$

$$= \frac{\pi}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \ldots\ldots \right\}$$

$$= \frac{\pi}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} \right)^2 k^{2n} \right\} \qquad [k^2 < 1].$$

If $$k' = \frac{1 - \sqrt{1 - k^2}}{1 + \sqrt{1 - k^2}}$$

$$K = \frac{\pi(1 + k')}{2} \left\{ 1 + \left(\frac{1}{2}\right)^2 k'^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k'^4 + \ldots \right\}$$

$$= \frac{\pi(1 + k')}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \ldots (2n - 1)}{2 \cdot 4 \cdot 6 \ldots\ldots 2n} \right)^2 k'^{2n} \right\}.$$

6.711 The complete elliptic integral of the second kind:

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta.$$

$$E = \frac{\pi}{2}\left\{1 - \left(\frac{1}{2}\right)^2 \frac{k^2}{1} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \dots\dots \right\}.$$

$$= \frac{\pi}{2}\left\{1 - \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots\dots 2n}\right)^2 \frac{k^{2n}}{2n-1}\right\}.$$

If $$k' = \frac{1 - \sqrt{1-k^2}}{1 + \sqrt{1-k^2}},$$

$$E = \frac{\pi(1-k')}{2}\left\{1 + 5\left(\frac{1}{2}\right)^2 k'^2 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k'^4 + \dots\right\}$$

$$= \frac{\pi(1-k')}{2}\left\{1 + \sum_{n=1}^{\infty}(4n+1)\left(\frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots 2n}\right)^2 k'^{2n}\right\}$$

$$= \frac{\pi}{2(1+k')}\left\{1 + \left(\frac{1}{2}\right)^2 k'^2 + \left(\frac{1}{2\cdot 4}\right)^2 k'^4 + \left(\frac{1\cdot 3}{2\cdot 4\cdot 6}\right)^2 k'^6 + \dots\right\}$$

$$= \frac{\pi}{2(1+k')}\left\{1 + k'^2\left[\frac{1}{4} + \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\dots\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n+2)}\right)^2 k'^{2n}\right]\right\}.$$

FOURIER'S SERIES

6.800 If $f(x)$ is uniformly convergent in the interval:

$$-c \leqq x \leqq +c$$

$$f(x) = \frac{1}{2}b_0 + b_1 \cos\frac{\pi x}{c} + b_2 \cos\frac{2\pi x}{c} + b_3 \cos\frac{3\pi x}{c} + \dots\dots$$
$$+ a_1 \sin\frac{\pi x}{c} + a_2 \sin\frac{2\pi x}{c} + a_3 \sin\frac{3\pi x}{c} + \dots\dots$$

$$b_m = \frac{1}{c}\int_{-c}^{+c} f(x)\cos\frac{m\pi x}{c}\,dx,$$

$$a_m = \frac{1}{c}\int_{-c}^{+c} f(x)\sin\frac{m\pi x}{c}\,dx.$$

6.801 If $f(x)$ is uniformly convergent in the interval:

$$0 \leqq x \leqq c$$

$$f(x) = \frac{1}{2}b_0 + b_1 \cos\frac{2\pi x}{c} + b_2 \cos\frac{4x\pi}{c} + b_3 \cos\frac{6\pi x}{c} + \dots\dots$$
$$+ a_1 \sin\frac{2\pi x}{c} + a_2 \sin\frac{4\pi x}{c} + a_3 \sin\frac{6\pi x}{c} + \dots\dots$$

$$b_m = \frac{2}{c}\int_0^c f(x)\cos\frac{2m\pi x}{c}\,dx,$$

$$a_m = \frac{2}{c}\int_0^c f(x)\sin\frac{2m\pi x}{c}\,dx.$$

6.802 Special Developments in Fourier's Series.

$$f(x) = a \text{ from } x = kc \text{ to } x = (k + \tfrac{1}{2})c,$$

$$f(x) = -a \text{ from } x = (k + \tfrac{1}{2})c \text{ to } x = (k + 1)c,$$

where k is any integer, including 0.

$$f(x) = \frac{4a}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{2(2n-1)\pi}{c} x.$$

6.803

$$\begin{aligned} f(x) &= \quad mx, & -\frac{c}{4} \leqslant x \leqslant +\frac{c}{4} \\ &= -m\left(x - \frac{c}{2}\right), & \frac{c}{4} \leqslant x \leqslant \frac{3c}{4} \\ &= \quad m\,(x - c), & \frac{3c}{4} \leqslant x \leqslant \frac{5c}{4} \\ &= -m\left(x - \frac{3c}{2}\right), & \frac{5c}{4} \leqslant x \leqslant \frac{7c}{4} \\ &\cdots\cdots \\ &\cdots\cdots \end{aligned}$$

$$f(x) = \frac{2mc}{\pi^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)^2} \sin \frac{2(2n-1)\pi}{c} x.$$

6.804

$$\begin{aligned} f(x) &= mx, & -\frac{c}{2} < x < +\frac{c}{2} \\ &= m(x - c), & +\frac{c}{2} < x < \frac{3c}{2}, \end{aligned}$$

$$f(x) = \frac{cm}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{2n\pi x}{c}.$$

6.805

$$\begin{aligned} f(x) &= -a, & -5b \leqslant x \leqslant -3b, \\ &= \quad \frac{a}{b}(x + 2b), & -3b \leqslant x \leqslant -b, \\ &= \quad a, & -b \leqslant x \leqslant +b, \\ &= -\frac{a}{b}(x - 2b), & b \leqslant x \leqslant 3b, \\ &= -a, & 3b \leqslant x \leqslant 5b. \\ &\cdots\cdots \\ &\cdots\cdots \end{aligned}$$

$$f(x) = \frac{8\sqrt{2}a}{\pi^2} \left\{ \cos \frac{\pi x}{4b} - \frac{1}{3^2} \cos \frac{3\pi x}{4b} - \frac{1}{5^2} \cos \frac{7\pi x}{4b} + \frac{1}{7^2} \cos \frac{7\pi x}{4b} + \cdots \right\}$$

6.806
$$f(x) = \frac{b}{l}x + b, \qquad -l \leqq x \leqq 0,$$
$$= -\frac{b}{l}x + b, \qquad 0 \leqq x \leqq l.$$
$$f(x) = \frac{8b}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\frac{\pi x}{2l}.$$

6.807
$$f(x) = \frac{a}{b}x, \qquad 0 \leqq x \leqq b,$$
$$= -\frac{a}{l-b}x + \frac{al}{l-b}, \qquad b \leqq x \leqq l,$$
$$f(x) = \frac{2al^2}{\pi^2 b(l-b)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi b}{l}\sin\frac{n\pi x}{l}.$$

6.810 $$x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin nx \qquad \left[-\pi < x < \pi\right].$$

6.811 $$\cos ax = \frac{2}{\pi}\sin a\pi\left\{\frac{1}{2a} + a\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2}\cos nx\right\} \qquad \left[-\pi < x < \pi\right].$$

6.812 $$\sin ax = \frac{2}{\pi}\sin a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2}\,n\sin nx \qquad \left[-\pi < x < \pi\right].$$

6.813 $$\frac{\pi - x}{2} = \sum_{n=1}^{\infty}\frac{\sin nx}{n} \qquad \left[0 < x < 2\pi\right].$$

6.814 $$\frac{1}{2}\log\frac{1}{2(1-\cos x)} = \sum_{n=1}^{\infty}\frac{\cos nx}{n} \qquad \left[0 < x < 2\pi\right].$$

6.815 $$\frac{\pi^2}{6} - \frac{\pi x}{2} + \frac{x^2}{4} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2} \qquad \left[0 < x < 2\pi\right].$$

6.816 $$\frac{\pi^2 x}{6} - \frac{\pi x^2}{4} + \frac{x^3}{12} = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3} \qquad \left[0 < x < 2\pi\right].$$

6.817 $$\frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi x^3}{12} - \frac{x^4}{48} = {\sum_{n=1}^{\infty}}'\frac{\cos nx}{n^4} \qquad \left[0 < x < 2\pi\right].$$

6.818 $$\frac{\pi^4 x}{90} - \frac{\pi^2 x^3}{36} + \frac{\pi x^4}{48} - \frac{x^5}{240} = \sum_{n=1}^{\infty}\frac{\sin nx}{n^5} \qquad \left[0 < x < 2\pi\right].$$

6.820 $x^2 = \frac{c^2}{3} - \frac{4c^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos\frac{n\pi x}{c}$ $\left[-c \leqslant x \leqslant c\right]$.

6.821 $\frac{e^x}{e^c - e^{-c}} = \frac{1}{2c} - c\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{(n\pi)^2 + c^2}\cos\frac{n\pi x}{c}$

$+ \pi\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{(n\pi)^2 + c^2}\sin\frac{n\pi x}{c}$ $\left[-c \leqslant x \leqslant c\right]$.

6.822 $e^{cx} = \frac{2c}{\pi}(e^{c\pi} - 1)\left\{\frac{1}{2c^2} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{c^2 + n^2} \cos nx\right\}$ $\left[0 < x < \pi\right]$.

6.823 $\cos 2x - \left(\frac{\pi}{2} - x\right)\sin 2x + \sin^2 x \log(4\sin^2 x) = \sum_{n=1}^{\infty} \frac{\cos 2(n+1)x}{n(n+1)}$

$[0 \leqslant x \leqslant \pi]$.

6.824 $\sin 2x - (\pi - 2x)\sin^2 x - \sin x \cos x \log(4\sin^2 x)$

$= \sum_{n=1}^{\infty} \frac{\sin 2(n+1)x}{n(n+1)}$ $\left[0 \leqslant x \leqslant \pi\right]$.

6.825 $\frac{1}{2} - \frac{\pi}{4}\sin x = \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n-1)(2n+1)}$ $\left[0 \leqslant x \leqslant \frac{\pi}{2}\right]$.

6.830 $\frac{r\sin x}{1 - 2r\cos x + r^2} = \sum_{n=1}^{\infty} r^n \sin nx$ $\left[r^2 < 1\right]$.

6.831 $\tan^{-1}\frac{r\sin x}{1 - r\cos x} = \sum_{n=1}^{\infty} \frac{1}{n} r^n \sin nx$ $\left[r < 1\right]$.

6.832 $\frac{1}{2}\tan^{-1}\frac{2r\sin x}{1 - r^2} = \sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1}\sin(2n-1)x$ $\left[r^2 < 1\right]$.

6.833 $\frac{1 - r\cos x}{1 - 2r\cos x + r^2} = \sum_{n=0}^{\infty} r^n \cos nx$ $\left[r^2 < 1\right]$.

6.834 $\log\frac{1}{\sqrt{1 - 2r\cos x + r^2}} = \sum_{n=1}^{\infty} \frac{1}{n} r^n \cos nx$ $\left[r^2 < 1\right]$.

6.835 $\frac{1}{2}\tan^{-1}\frac{2r\cos x}{1-r^2}=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{r^{2n-1}}{2n-1}\cos(2n-1)x \qquad \left[r^2<1\right].$

NUMERICAL SERIES

6.900

$$S_n=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\dots\dots=\sum_{k=1}^{\infty}\frac{1}{k^n},$$

$S_1=\infty$

$S_2=\frac{\pi^2}{6}=1.6449340668$

$S_3=\frac{\pi^3}{25.79436\dots}=1.2020569032$

$S_4=\frac{\pi^4}{90}=1.0823232337$

$S_5=\frac{\pi^5}{295.1215}=1.0369277551$

$S_6=\frac{\pi^6}{945}=1.0173430620,$

$S_7=\frac{\pi^7}{2995.286\dots}=1.0083492774$

$S_8=\frac{\pi^8}{9450}=1.0040773562,$

$S_9=\frac{\pi^9}{29749.35\dots}=1.0020083928,$

$S_{10}=1.0009945751,$

$S_{11}=1.0004941886.$

6.901

$$u_n=1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\dots\dots=\sum_{k=0}^{\infty}(-1)^{k+1}\frac{1}{(2k+1)^n},$$

$u_1=\frac{\pi}{4},$

$u_2=0.9159656\dots$

$u_4=0.98894455\dots.$

$u_6=0.99868522\dots.$

A table of u_n from $n=1$ to $n=38$ to 18 decimal places is given by Glaisher, Messenger of Mathematics, 42, p. 49, 1913.

6.902 Bernoulli's Numbers.

1. $\frac{2^{2n-1}\pi^{2n}}{(2n)!}B_n=\frac{1}{1^{2n}}+\frac{1}{2^{2n}}+\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\dots\dots=\sum_{k=1}^{\infty}\frac{1}{k^{2n}}.$

2. $\frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}B_n=\frac{1}{1^{2n}}+\frac{1}{3^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}+\dots\dots=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{2n}}.$

3. $\frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}B_n=\frac{1}{1^{2n}}-\frac{1}{2^{2n}}+\frac{1}{3^{2n}}-\frac{1}{4^{2n}}+\dots\dots=\sum_{k=1}^{\infty}(-1)^{n-1}\frac{1}{k^{2n}}.$

$B_1=\frac{1}{6},\qquad B_3=\frac{1}{42},$

$B_2=\frac{1}{[illegible]},\qquad B_4=\frac{1}{[illegible]},$

$$B_5 = \frac{5}{66}, \qquad B_8 = \frac{3617}{510},$$

$$B_6 = \frac{691}{2730}, \qquad B_9 = \frac{43867}{798},$$

$$B_7 = \frac{7}{6}, \qquad B_{10} = \frac{174611}{330}.$$

6.903 Euler's Numbers

$$\frac{\pi^{2n+1}}{2^{2n+2}(2n)!} E_n = 1 - \frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \quad \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{(2k-1)^{2n+1}}.$$

$$E_1 = 1, \qquad E_4 = 1385,$$

$$E_2 = 5, \qquad E_5 = 50521,$$

$$E_3 = 61, \qquad E_6 = 2702765.$$

6.904

$$E_n - \frac{2n(2n-1)}{2!} E_{n-1} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!} E_{n-2} - \ldots \ldots - \ldots \ldots + (-1)^n = 0.$$

6.905

$$\frac{2^{2n}(2^{2n}-1)}{2n} B_n = (2n-1)E_{n-1} - \frac{(2n-1)(2n-2)(2n-3)}{3!} E_{n-2}$$

$$+ \frac{(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)}{5!} E_{n-3} - \ldots \ldots + (-1)^{n-1}.$$

6.910

$$S_r = \sum_{n=1}^{\infty} \frac{n^r}{n!}$$

$$S_1 = e, \qquad S_5 = 52e,$$

$$S_2 = 2e, \qquad S_6 = 203e,$$

$$S_3 = 5e, \qquad S_7 = 877e,$$

$$S_4 = 15e, \qquad S_8 = 4140e.$$

6.911

$$S_r = \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)^r}.$$

$$S_1 = \frac{1}{2}, \qquad S_3 = \frac{32 - 3\pi^2}{64},$$

$$S_2 = \frac{\pi^2 - 8}{16}, \qquad S_4 = \frac{\pi^4 + 30\pi^2 - 384}{768}.$$

6.912

1. $\log 2 = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n}$.

2. $\frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2 = \sum_{n=1}^{\infty} \frac{1}{n^2 2^n}$.

6.913

1. $2\log 2 - 1 = \sum_{n=1}^{\infty} \frac{1}{n(4n^2 - 1)}$.

2. $\frac{3}{2}(\log 3 - 1) = \sum_{n=1}^{\infty} \frac{1}{n(9n^2 - 1)}$.

3. $-3 + \frac{3}{2}\log 3 + 2\log 2 = \sum_{n=1}^{\infty} \frac{1}{n(36n^2 - 1)}$.

6.914

$$S_r = \sum_{n=1}^{\infty} \left(\frac{1 \cdot 3 \cdot 5 \cdot \ldots \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \ldots 2n}\right)^2 \frac{1}{2n + r},$$

$u_2 = 0.9159656\ldots$ (see 6.901)

$S_0 = 2\log 2 - \frac{4}{\pi}u_2,$

$S_1 = \frac{4}{\pi}u_2 - 1,$

$S_2 = \frac{2}{\pi} - \frac{1}{2},$

$S_3 = \frac{1}{2\pi}(2u_2 + 1) - \frac{1}{3},$

$S_4 = \frac{10}{9\pi} - \frac{1}{4},$

$S_5 = \frac{1}{32\pi}(18u_2 + 13) - \frac{1}{5},$

$S_6 = \frac{178}{225\pi} - \frac{1}{6},$

$S_7 = \frac{1}{128\pi}(50u_2 + 43) - \frac{1}{7},$

$S_{-1} = 1 - \frac{2}{\pi},$

$S_{-2} = \frac{1}{2}\log 2 + \frac{1}{4} - \frac{1}{2\pi}(2u_2 + 1),$

$S_{-3} = \frac{1}{3} - \frac{10}{9\pi},$

$S_{-4} = \frac{9}{32}\log 2 + \frac{11}{128} - \frac{1}{32\pi}(18u_2 + 13),$

$S_{-5} = \frac{1}{5} - \frac{178}{225\pi},$

$S_{-6} = \frac{25}{128}\log 2 + \frac{71}{1536} - \frac{1}{128\pi}(50u_2 + 43).$

When r is a negative even integer the value $n = \frac{r}{2}$ is to be excluded in the summation.

.915

$$\frac{1 \cdot 3 \cdot 5 \cdot \ldots \ldots (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \ldots 2n} = \frac{(2n-1)!}{2^{2n-1} n!(n-1)!}.$$

∞ 1

3. $\dfrac{\pi}{2} - 1 = \sum_{n=1}^{\infty} A_n \dfrac{1}{2n+1}.$

4. $\log(1+\sqrt{2}) - 1 = \sum_{n=1}^{\infty} (-1)^n A_n \dfrac{1}{2n+1}.$

5. $\dfrac{1}{2} = \sum_{n=1}^{\infty} A_n^2 \dfrac{4n+1}{(2n-1)(2n+2)}.$

6. $\dfrac{2}{\pi} - \dfrac{1}{2} = \sum_{n=1}^{\infty} (-1)^{n+1} A_n^3 \dfrac{4n+1}{(2n-1)(2n+2)}.$

7. $\dfrac{2}{\pi} - 1 = \sum_{n=1}^{\infty} (-1)^n A_n^3 (4n+1).$

8. $\dfrac{1}{2} - \dfrac{4}{\pi^2} = \sum_{n=1}^{\infty} A_n^4 \dfrac{4n+1}{(2n-1)(2n+2)}.$

6.916

If m is an integer, and $n = m$ is excluded from the summation:

1. $-\dfrac{3}{4m^2} = \sum_{n=1}^{\infty} \dfrac{1}{m^2 - n^2}.$

2. $\dfrac{3}{4m^2} = \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{m^2 - n^2}.$ (m even)

6.917

1. $1 = \sum_{n=2}^{\infty} \dfrac{n-1}{n!}.$

2. $\dfrac{1}{2} = \sum_{n=1}^{\infty} \dfrac{1}{4n^2 - 1}.$

3. $2 \log 2 = \sum_{n=1}^{\infty} \dfrac{12n^2 - 1}{n(4n^2 - 1)^2}.$

6.918 $\dfrac{2}{\sqrt{3}} \log \dfrac{1+\sqrt{3}}{\sqrt{2}} = 1 + \sum_{n=1}^{\infty} (-1)^n \dfrac{2\cdot 4\cdot 6 \cdot \ldots \cdot 2n}{3\cdot 5\cdot 7 \cdot \ldots \cdot (2n+1)} \dfrac{1}{2^n}.$

6.919 $\dfrac{1}{2}(1 - \log 2) = \sum_{n=1}^{\infty} \left\{ n \log\left(\dfrac{2n+1}{2n-1}\right) - 1 \right\}.$

6.920

2. $\frac{1}{e} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \dots \quad = 0.36788.$

3. $\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots = 1.54308.$

4. $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \dots = 1.175201.$

5. $\cos 1 = 1 - \frac{1}{2!} + \frac{1}{4!} - \dots \quad = 0.54030.$

6. $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \dots \quad = 0.84147.$

6.921

1. $\frac{4}{5} = 1 - \frac{1}{2^2} + \frac{1}{2^4} - \frac{1}{2^6} + \dots$

2. $\frac{9}{10} = 1 - \frac{1}{3^2} + \frac{1}{3^4} - \frac{1}{3^6} + \dots$

3. $\frac{16}{17} = 1 - \frac{1}{4^2} + \frac{1}{4^4} - \frac{1}{4^6} + \dots$

4. $\frac{25}{26} = 1 - \frac{1}{5^2} + \frac{1}{5^4} - \frac{1}{5^6} + \dots$

6.922 $\quad \frac{(2^{\frac{1}{4}} - 1)\Gamma(\frac{1}{4})}{2^{\frac{11}{4}}\pi^{\frac{3}{4}}} = e^{-\pi} + e^{-9\pi} + e^{-25\pi} + \dots;\ \Gamma(\tfrac{1}{4}) = 3.6256\dots$

6.923 (Special cases of **6.705**):

1. $\frac{1}{1\cdot2\cdot3} + \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} + \dots \quad = \log 2 - \frac{1}{2}.$

2. $\frac{1}{1\cdot2\cdot3} - \frac{1}{3\cdot4\cdot5} + \frac{1}{5\cdot6\cdot7} - \dots \quad = \frac{1}{2}(1 - \log 2).$

3. $\frac{1}{2\cdot3\cdot4} + \frac{1}{4\cdot5\cdot6} + \frac{1}{6\cdot7\cdot8} + \dots \quad = \frac{3}{4} - \log 2.$

4. $\frac{1}{2\cdot3\cdot4} - \frac{1}{4\cdot5\cdot6} + \frac{1}{6\cdot7\cdot8} - \dots \quad = \frac{1}{4}(\pi - 3).$

5. $\frac{1}{1\cdot2\cdot3} + \frac{1}{4\cdot5\cdot6} + \frac{1}{7\cdot8\cdot9} + \dots \quad = \frac{1}{4}\left(\frac{\pi}{\sqrt{3}} - \log 3\right).$

6. $\frac{1}{2\cdot3\cdot4} + \frac{1}{6\cdot7\cdot8} + \frac{1}{10\cdot11\cdot12} + \dots \quad = \frac{\pi}{8} - \frac{1}{2}\log 2.$

7. $\frac{1}{1\cdot2\cdot3\cdot4} + \frac{1}{4\cdot5\cdot6\cdot7} + \frac{1}{7\cdot8\cdot9\cdot10} + \dots = \frac{1}{6}\left(1 + \frac{\pi}{2\sqrt{3}}\right) - \frac{1}{4}\log 3.$

VII. SPECIAL APPLICATIONS OF ANALYSIS.

7.10 Indeterminate Forms.

7.101 $\frac{0}{0}$. If $\frac{f(x)}{F(x)}$ assumes the indeterminate value $\frac{0}{0}$ for $x = a$, the true value of the quotient may be found by replacing $f(x)$ and $F(x)$ by their developments in series, if valid for $x = a$.

Example:

$$\left[\frac{\sin^2 x}{1 - \cos x}\right]_{x=0};$$

$$\frac{\sin^2 x}{1 - \cos x} = \frac{\left(x - \frac{x^3}{3!} + \ldots\right)^2}{\frac{x^2}{2!} - \frac{x^4}{4!} + \ldots} = \frac{\left(1 - \frac{x^2}{3!} + \ldots\right)^2}{\frac{1}{2!} - \frac{x^2}{4!} + \ldots}$$

Therefore,

$$\left[\frac{\sin^2 x}{1 - \cos x}\right]_{x=0} = 2.$$

7.102 L'Hospital's Rule. If $f(a + h)$ and $F(a + h)$ can be developed by Taylor's Theorem (**6.100**) then the true value of $\frac{f(x)}{F(x)}$ for $x = a$ is,

$$\frac{f'(a)}{F'(a)}$$

provided that this has a definite value (0, finite, or infinite). If the ratio of the first derivatives is still indeterminate, the true value may be found by taking that of the ratio of the first one of the higher derivatives that is definite.

7.103 The true value of $\frac{f(x)}{F(x)}$ for $x = a$ is the limit, for $h = 0$, of

$$\frac{q!}{p!} h^{p-q} \frac{f^{(p)}(a)}{F^{(q)}(a)}$$

where $f^{(p)}(a)$ and $F^{(q)}(a)$ are the first of the higher derivatives of $f(x)$ and $F(x)$ that do not vanish for $x = a$. The true value of $\frac{f(x)}{F(x)}$ for $x = a$ is 0 if $p > q$, ∞ if $p < q$, and equal to $\frac{f^{(p)}(a)}{F^{(p)}(a)}$ if $p = q$.

Example:

$$\left[\frac{\sinh x - x\cosh x}{\sin x - x\cos x}\right]_{x=0} = \left[\frac{-x\sinh x}{x\sin x}\right]_{x=0}$$

$$= \left[-\frac{\sinh x}{\sin x}\right]_{x=0} = \left[-\frac{\cosh x}{\cos x}\right]_{x=0} = -1.$$

7.104 Failure of L'Hospital's Rule. In certain cases this rule fails to determine the true value of an expression for the reason that all the higher derivatives vanish at the limit. In such cases the true value may often be found by factoring the given expression, or resolving into partial fractions (**1.61**).

Example:

$$\left[\frac{\sqrt{x^2 - a^2}}{\sqrt{x - a}}\right]_{x=a} = \left[\sqrt{x + a}\right]_{x=a} = \sqrt{2a}.$$

7.105 In applying L'Hospital's Rule, if any of the successive quotients contains a factor which can be evaluated at once its determinate value may be substituted.

Example:

$$\left[\frac{(1 - x)e^x - 1}{\tan^2 x}\right]_{x=0} = \left[\frac{-xe^x}{2\tan x\sec^2 x}\right]_{x=0}$$

$$\left[\frac{x}{\tan x}\right]_{x=0} = 1.$$

Hence the given function is,

$$\left[-\frac{e^x}{2\sec^2 x}\right]_{x=0} = -\frac{1}{2}.$$

7.106 If the given function can be separated into factors each of which is indeterminate, the factors may be evaluated separately.

Example:

$$\left[\frac{(e^x - 1)\tan^2 x}{x^3}\right]_{x=0} = \left[\left(\frac{\tan x}{x}\right)^2 \frac{e^x - 1}{x}\right]_{x=0} = 1.$$

7.110 $\frac{\infty}{\infty}$. If, for $x = a$, $\frac{f(x)}{F(x)}$ takes the form $\frac{\infty}{\infty}$, this quotient may be written:

$$\frac{\dfrac{1}{F(x)}}{\dfrac{1}{f(x)}}$$

which takes the form $\frac{0}{0}$ for $x = a$ and the preceding sections will apply to it.

7.111 L'Hospital's Rule (**7.102**) may be applied directly to indeterminate forms $\frac{\infty}{\infty}$, if the expansion by Taylor's Theorem is valid.

Example:

$$\left[\frac{x}{e^x}\right]_{x=\infty} = \left[\frac{1}{e^x}\right]_{x=\infty} = 0.$$

7.112 If $f(x)$ and x approach ∞ together, and if $f(x+1) - f(x)$ approaches a definite limit, then,

$$\underset{x\to\infty}{\text{Limit}}\left[\frac{f(x)}{x}\right] = \underset{x\to\infty}{\text{Limit}}\left[f(x+1) - f(x)\right].$$

7.120 $0 \times \infty$ . . If, for $x = a$, $f(x) \times F(x)$ takes the form $0 \times \infty$, this product may be written,

$$\frac{\dfrac{f(x)}{1}}{\dfrac{1}{F(x)}}$$

which takes the form $\frac{0}{0}$ (7.101).

7.130 $\infty - \infty$. If, $\underset{x\to a}{\text{Limit}}\ f(x) = \infty$ and $\underset{x\to\infty}{\text{Limit}}\ F(x) = \infty$,

$$f(x) - F(x) = f(x)\left\{1 - \frac{F(x)}{f(x)}\right\}.$$

If $\underset{x\to\infty}{\text{Limit}}\ \dfrac{F(x)}{f(x)}$ is different from unity the true value of $f(x) - F(x)$ for $x = a$ is ∞.

If $\underset{x\to\infty}{\text{Limit}}\ \dfrac{F(x)}{f(x)} = +1$, the expression has the indeterminate form $\infty \times 0$ which may be treated by **7.120**.

7.140 $1^\infty, 0^0, \infty^0$. If $\{F(x)\}^{f(x)}$ is indeterminate in any of these forms for $x = a$, its true value may be found by finding the true value of the logarithm of the given expression.

Example:

$$\left[\left(\frac{1}{x}\right)^{\tan x}\right]_{x\to 0}.$$

$$\left(\frac{1}{x}\right)^{\tan x} = y; \quad \log y = -\tan x \cdot \log x,$$

$$\left[\tan x\cdot\log x\right]_{x=0} = \left[\frac{\log x}{\cot x}\right]_{x=0} = \left[\frac{\frac{1}{x}}{-\csc^2 x}\right]_{x=0} = \left[-\frac{\sin x}{x}\cdot\sin x\right]_{x=0} = 0.$$

Hence,

$$\left[\left(\frac{1}{x}\right)^{\tan x}\right]_{x=0} = 1.$$

7.141 If $f(x)$ and x approach ∞ together, and $\dfrac{f(x+1)}{f(x)}$ approaches a definite limit, then,

$$\underset{x\to\infty}{\text{Limit}}\left[\{f(x)\}^{\frac{1}{x}}\right] = \underset{x\to\infty}{\text{Limit}}\ \frac{f(x+1)}{f(x)}.$$

7.150 Differential Coefficients of the form $\frac{0}{0}$. In determining the differential coefficient $\dfrac{dy}{dx}$ from an equation $f(x, y) = 0$, by means of the formula,

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} \tag{1}$$

it may happen that for a pair of values, $x = a$, $y = b$, satisfying $f(x, y) = 0$, $\dfrac{dy}{dx}$ takes the form $\frac{0}{0}$.

Writing $\dfrac{dy}{dx} = y'$, and applying **7.102** to the quotient (1), a quadratic equation is obtained for determining y', giving, in general, two different determinate values. If y' is still indeterminate, apply **7.102** again, giving a cubic equation for determining y'. This process may be continued until determinate values result.

Example:

$$f(x, y) = (x^2 + y^2)^2 - c^2xy = 0,$$

$$y' = -\frac{4x(x^2 + y^2) - c^2y}{4y(x^2 + y^2) - c^2x}.$$

For $x = 0$, $y = 0$, y' takes the value $\frac{0}{0}$.

Applying **7.102**,

$$-y' = \frac{12x^2 + 4y^2 + (8xy - c^2)y'}{4y'(x^2 + 3y^2) + 8xy - c^2}.$$

Solving this quadratic equation in y', the two determinate values, $y' = 0$, $y' = \infty$, result for $x = 0$, $y = 0$.

7.17 Special Indeterminate Forms and Limiting Values. In the following the notation $[f(x)]_a$ means the limit approached by $f(x)$ as x approaches a as a limit.

7.171

1. $\left[\left(1+\frac{c}{x}\right)^x\right]_\infty = e^c$ (c a constant).

2. $[\sqrt{x+c}-\sqrt{x}]_\infty = 0.$

3. $[\sqrt{x(x+c)}-x]_\infty = \frac{c}{2}.$

4. $[\sqrt{(x+c_1)(x+c_2)}-x]_\infty = \frac{1}{2}(c_1+c_2).$

5. $\left[\sqrt[n]{(x+c_1)(x+c_2)\ldots\ldots(x+c_n)}-x\right]_\infty = \frac{1}{n}(c_1+c_2+\ldots c_n).$

6. $\left[\frac{\log(c_1+c_2e^x)}{x}\right]_\infty = 1.$

7. $\left[\log\left(c_1+c_2e^x\right)\cdot\log\left(1+\frac{1}{x}\right)\right]_\infty = 1.$

8. $\left[\left(\frac{\log x}{x}\right)^{\frac{1}{x}}\right]_\infty = 1.$

9. $\left[\frac{x}{(\log x)^m}\right]_\infty = \infty.$

10. $\left[\frac{a^x}{x^m}\right]_\infty = \infty$ $(a>1).$

11. $\left[\frac{a^x}{x!}\right]_\infty = 0$ (x a positive integer).

12. $\left[x^{\frac{1}{x}}\right]_\infty = 1.$

13. $\left[\frac{\log x}{x}\right]_\infty = 0.$

14. $\left[(a+bc^x)^{\frac{1}{x}}\right]_\infty = c$ $(c>1).$

15. $\left[\left(\frac{1}{a+bc^x}\right)^{\frac{c}{x}}\right]_\infty = e^{-c}.$

16. $\left[\frac{x}{\alpha+\beta x^2}\cdot\log(a+be^x)\right]_\infty = \frac{1}{\beta}.$

17. $\left[\left(a+bx^m\right)^{\frac{1}{\alpha+\beta\log x}}\right]_\infty = e^{\frac{m}{\beta}}$ $(m>0).$

.172

1. $\left[x \sin \frac{c}{x}\right]_{\infty} = c.$

2. $\left[x\left(1 - \cos \frac{c}{x}\right)\right]_{\infty} = 0.$

3. $\left[x^2\left(1 - \cos \frac{c}{x}\right)\right]_{\infty} = \frac{c^2}{2}.$

4. $\left[\left(\cos \frac{c}{x}\right)^{x}\right]_{\infty} = 1.$

5. $\left[\left(\cos \frac{c}{x}\right)^{x^2}\right]_{\infty} = e^{-\frac{c^2}{2}}.$

6. $\left[\left(\frac{\sin \frac{c}{x}}{\frac{c}{x}}\right)^{x}\right]_{\infty} = 1.$

7. $\left[\frac{\cot \frac{c}{x}}{x}\right]_{\infty} = \frac{1}{c}.$

8. $\left[\sin \frac{c}{x} \cdot \log (a + be^{x})\right]_{\infty} = c.$

9. $\left[\left(\cos \sqrt{\frac{2c}{x}}\right)^{x}\right]_{\infty} = e^{-c}.$

10. $\left[\left(1 + a \tan \frac{c}{x}\right)^{x}\right]_{\infty} = e^{ac}.$

11. $\left[\left(\cos \frac{c}{x} + a \sin \frac{c}{x}\right)^{x}\right]_{\infty} = e^{ac}.$

7.173

1. $\left[\frac{\sin x}{x}\right]_0 = 1.$

2. $\left[\frac{\tan x}{x}\right]_0 = 1.$

3. $\left[\left(\frac{\sin nx}{x}\right)^{m}\right]_0 = n^{m}.$

4. $\left[\sin^{-1} x \cdot \cot x\right]_0 = 1.$

5. $\left[\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}^{\cot x}\right]_0 = e.$

7.174

1. $\left[x^{x}\right]_0 = 1.$

2. $\left[x^{\frac{1}{a + b \log x}}\right]_0 = e^{\frac{1}{b}}.$

3. $\left[x^{\frac{1}{\log (e^{x} - 1)}}\right]_0 = e.$

4. $\left[x^{m} \log \frac{1}{x}\right]_0 = 0 \qquad (m \geqq 1).$

5. $\left[\log \cos x \cdot \cot x\right]_0 = 0.$

6. $\left[\log \tan \left(\frac{\pi}{4} + \frac{x}{2}\right) \cdot \cot x\right]_0 = 1.$

7. $\left[\frac{e^{x} - 1}{x}\right]_0 = 1.$

8. $\left[x^{m} \log x\right]_0 = 0 \qquad (m > 0).$

9. $\left[\frac{e^{x} - e^{-x} - 2x}{(e^{x} - 1)^{3}}\right]_0 = \frac{1}{3}.$

10. $\left[xe^{\frac{1}{x}}\right]_0 = \infty.$

11. $\left[\frac{e^{x} - e^{-x}}{\log (1 + x)}\right]_0 = 2.$

12. $\left[\frac{\log \tan 2x}{\log \tan x}\right]_0 = 1.$

7.175

1. $\left[x^{\frac{1}{1-x}}\right]_1 = \frac{1}{e}.$

2. $\left[(\pi - 2x)\tan x\right]_{\frac{\pi}{2}} = 2.$

3. $\left[\log\left(2 - \frac{x}{c}\right)\cdot\tan\frac{\pi x}{2c}\right]_c = \frac{2}{\pi}.$

4. $\left[(e^a - e^x)\tan\frac{\pi x}{2a}\right]_a = \frac{2a}{\pi}e^a.$

5. $\left[\cos^{-1}\frac{x}{c}\cdot\tan\frac{\pi x}{2c}\right]_c = \infty$

6. $\left[(a + be^{\tan x})^{\pi - 2x}\right]_{\frac{\pi}{2}} = e^2.$

7. $\left[\left(2 - \frac{2x}{\pi}\right)^{\tan x}\right]_{\frac{\pi}{2}} = e^{\frac{2}{\pi}}$

8. $\left[(\tan x)^{\tan 2x}\right]_{\frac{\pi}{4}} = \frac{1}{e}.$

7.18 Limiting Values of Sums.

1. $$\operatorname*{Limit}_{n\to\infty}\left(\frac{1^k + 2^k + 3^k + \ldots\ldots + n^k}{n^{k+1}}\right) = \frac{1}{k+1} \text{ if } k > -1.$$
$$\infty \text{ if } k < -1.$$

2. $$\operatorname*{Limit}_{n\to\infty}\left(\frac{1}{na} + \frac{1}{na+b} + \frac{1}{na+2b} + \ldots\ldots + \frac{1}{na+(n-1)b}\right)$$
$$= \frac{\log(a+b) - \log a}{b} \quad (a, b > 0).$$

3. $$\operatorname*{Limit}_{n\to\infty}\left(\frac{n-1^2}{1\cdot 2\cdot(n+1)} + \frac{n-2^2}{2\cdot 3\cdot(n+2)} + \frac{n-3^2}{3\cdot 4\cdot(n+3)} + \ldots\ldots\right.$$
$$\left. + \frac{(n-n^2}{n\cdot(n+1)\cdot(n+n)}\right) = 1 - \log 2.$$

4. $$\operatorname*{Limit}_{n\to\infty}\left[\left(a + b\frac{\sqrt{1}}{n}\right)^2 + \left(a^2 + b\frac{\sqrt{2}}{n}\right)^2 + \left(a^3 + b\frac{\sqrt{3}}{n}\right)^2 + \ldots\ldots\right.$$
$$\left. + \left(a^n + b\frac{\sqrt{n}}{n}\right)^2\right] = \frac{a^2}{1-a^2} + \frac{b^2}{2},$$
if a is a positive proper fraction.

5. $$\operatorname*{Limit}_{n\to\infty}\left[\sqrt{a + \frac{b}{n}} + \sqrt{a^2 + \frac{b}{n}} + \sqrt{a^3 + \frac{b}{n}} + \ldots + \sqrt{a^n + \frac{b}{n}}\right] = \infty,$$
if $b > 0$ and a is a positive proper fraction.

6. $$\operatorname*{Limit}_{n\to\infty}\left[\sqrt{a + \frac{b}{1\cdot n}} + \sqrt{a^2 + \frac{b}{2\cdot n}} + \sqrt{a^3 + \frac{b}{3\cdot n}} + \ldots\ldots + \sqrt{a^n + \frac{b}{n\cdot n}}\right]$$
$$= \frac{\sqrt{a}}{1 - \sqrt{a}} + 2\sqrt{b},$$
if $b > 0$ and a is a positive proper fraction.

7. $$\operatorname*{Limit}_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n\right] = \gamma = 0.5772157\ldots$$
(6.602).

7.19 Limiting Values of Products.

1. $$\underset{n\to\infty}{\text{Limit}}\left[\left(1+\frac{c}{n}\right)\left(1+\frac{c}{n+1}\right)\left(1+\frac{c}{n+2}\right)\cdots\cdots\left(1+\frac{c}{2n-1}\right)\right]=2^c,$$
if $c>0$.

2. $$\underset{n\to\infty}{\text{Limit}}\left[\left(1+\frac{c}{na}\right)\left(1+\frac{c}{na+b}\right)\left(1+\frac{c}{na+2b}\right)\cdots\cdots\left(1+\frac{c}{na+(n-1)b}\right)\right]$$
$$=\left(1+\frac{b}{a}\right)^{\frac{c}{b}},$$
if a, b, c are all positive.

3. $$\underset{n\to\infty}{\text{Limit}}\left[\frac{\{m(m+1)\,(m+2)\cdots\cdots(m+n-1)\}^{\frac{1}{n}}}{m+\frac{1}{2}(n-1)}\right]=\frac{2}{e},$$
if $m>0$.

4. $$\underset{n\to}{\text{Limit}}\left[\left(1+\frac{2c}{n^2}\right)\left(1+\frac{4c}{n^2}\right)\left(1+\frac{6c}{n^2}\right)\cdots\cdots\left(1+\frac{2nc}{n^2}\right)\right]=e^c.$$

7.20 Maxima and Minima.

7.201 Functions of One Variable. $y=f(x)$ is a maximum or minimum for the values of x satisfying the equation, $f'(x)=\dfrac{\partial f(x)}{\partial x}=0$, provided that $f'(x)$ is continuous for these values of x.

7.202 If, for $x=a$, $f'(a)=0$,

$y=f(a)$ is a maximum if $f''(a)<0$
$y=f(a)$ is a minimum if $f''(a)>0$.

Example:

$$y=\frac{x}{x^2+\alpha x+\beta},\qquad \beta>0,$$

$$f'(x)=\frac{-x^2+\beta}{(x^2+\alpha x+\beta)^2},$$

$$f'(x)=0 \text{ when } x=\pm\sqrt{\beta},$$

$$f''(x)=\frac{2x^3-6\beta x-2\alpha\beta}{(x^2+\alpha x+\beta)^3}$$

For $x=+\sqrt{\beta}$, $f''(x)=\dfrac{-2}{\sqrt{\beta}}\,\dfrac{1}{(2\sqrt{\beta}+\alpha)^2}$ Maximum,

For $x = -\sqrt{\beta}$, $f''(x) = \dfrac{+2}{\sqrt{\beta}} \dfrac{1}{(2\sqrt{\beta} - \alpha)^2}$ Minimum,

$$y_{max} = \frac{1}{\alpha + 2\sqrt{\beta}},$$

$$y_{min} = \frac{1}{\alpha - 2\sqrt{\beta}}.$$

7.203 If for $x = a$, $f'(a) = 0$ and $f''(a) = 0$, in order to determine whether $y = f(a)$ is a maximum or minimum it is necessary to form the higher differential coefficients, until one of even order is found which does not vanish for $x = a$. $y = f(a)$ is a maximum or minimum according as the first of the differential coefficients, $f''(a)$, $f^{iv}(a)$, $f^{vi}(a)$, ... of even order which does not vanish is negative or positive.

7.210 Functions of Two Variables. $F(x, y)$ is a maximum or minimum for the pair of values of x and y that satisfy the equations,

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0,$$

and for which

$$\left(\frac{\partial^2 F}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} < 0.$$

If both $\dfrac{\partial^2 F}{\partial x^2}$ and $\dfrac{\partial^2 F}{\partial y^2}$ are negative for this pair of values of x and y, $F(x, y)$ is a maximum. If they are both positive $F(x, y)$ is a minimum.

7.220 Functions of n Variables. For the maximum or minimum of a function of n variables, $F(x_1, x_2 \ldots\ldots, x_n)$, it is necessary that the first derivatives, $\dfrac{\partial F}{\partial x_1}, \dfrac{\partial F}{\partial x_2}, \ldots\ldots, \dfrac{\partial F}{\partial x_n}$ all vanish; and that the lowest order of the higher derivatives which do not all vanish be an even number. If this number be 2 the necessary condition for a minimum is that all of the determinants,

$$D_k = \begin{vmatrix} f_{11} & f_{12} & \ldots\ldots & f_{1k} \\ f_{21} & f_{22} & \ldots\ldots & f_{2k} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ f_{k1} & f_{k2} & \ldots\ldots & f_{kk} \end{vmatrix}, \quad k = 1, 2, \ldots, n,$$

where

$$f_{ij} = \frac{\partial^2 F}{\partial x_i\, \partial x_j},$$

shall be positive. For a maximum the determinants must be alternately negative and positive, beginning with $D_1 = \frac{\partial^2 F}{\partial x_1^2}$ negative.

7.230 Maxima and Minima with Conditions. If $F(x_1, x_2, \ldots\ldots, x_n)$ is to be made a maximum or minimum subject to the conditions,

$$\text{1.}\quad \begin{cases} \phi_1(x_1, x_2, \ldots\ldots, x_n) = 0 \\ \phi_2(x_1, x_2, \ldots\ldots, x_n) = 0 \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \phi_k(x_1, x_2, \ldots\ldots, x_n) = 0, \end{cases}$$

where $k<n$, the necessary conditions are,

$$\text{2.}\qquad \frac{\partial F}{\partial x_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial \phi_j}{\partial x_i} = 0 \qquad i = 1, 2, \ldots, n,$$

where the λ's are k undetermined multipliers. The n equations (2) together with the k equations of condition (1) furnish $k+n$ equations to determine the $k+n$ quantities, $x_1, x_2, \ldots\ldots, x_n, \lambda_1, \lambda_2, \ldots\ldots, \lambda_k$.

Example:

To find the axes of the ellipsoid, referred to its center as origin,

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz = 1.$$

Denoting the radius vector to the surface by r, and its direction-cosines by l, m, n, so that $x = lr$, $y = mr$, $z = nr$, it is necessary to find the maxima and minima of

$$r^2 = \frac{1}{a_{11}l^2 + a_{22}m^2 + a_{33}n^2 + 2a_{12}lm + 2a_{23}mn + 2a_{13}ln},$$

subject to the condition

$$\phi(l, m, n) = l^2 + m^2 + n^2 - 1 = 0.$$

This is the same as finding the minima and maxima of

$$F(l, m, n) = a_{11}l^2 + a_{22}m^2 + a_{33}n^2 + 2a_{12}lm + 2a_{23}mn + 2a_{13}ln.$$

Equation (2) gives:

$$(a_{11} + \lambda)l + a_{12}m + a_{13}n = 0,$$
$$a_{12}l + (a_{22} + \lambda)m + a_{23}n = 0,$$
$$a_{13}l + a_{23}m + (a_{33} + \lambda)n = 0.$$

Multiplying these 3 equations by l, m, n respectively and adding,

$$\lambda = -\frac{1}{r^2}.$$

Then by (1. 1.303) the 3 values of r are given by the 3 roots of

$$\begin{vmatrix} a_{11} - \frac{1}{r^2} & a_{12} & a_{13} \\ a_{12} & a_{22} - \frac{1}{r^2} & a_{23} \\ a_{13} & a_{23} & a_{33} - \frac{1}{r^2} \end{vmatrix} = 0.$$

7.30 Derivatives.

7.31 First Derivatives.

1. $\dfrac{dx^n}{dx} = nx^{n-1}$.

2. $\dfrac{da^x}{dx} = a^x \log a$.

3. $\dfrac{de^x}{dx} = e^x$.

4. $\dfrac{dx^x}{dx} = x^x(1 + \log x)$.

5. $\dfrac{d \log_a x}{dx} = \dfrac{1}{x \log a} = \dfrac{\log_a e}{x}$.

6. $\dfrac{d \log x}{dx} = \dfrac{1}{x}$.

7. $\dfrac{dx^{\log x}}{dx} = 2x^{\log x - 1} \log x$.

8. $\dfrac{d(\log x)^x}{dx} = (\log x)^{x-1} \{1 + \log x \cdot \log \log x\}$.

9. $\dfrac{d\left(\dfrac{x}{e}\right)^x}{dx} = \left(\dfrac{x}{e}\right)^x \log x$.

10. $\dfrac{d \sin x}{dx} = \cos x$.

11. $\dfrac{d \cos x}{dx} = -\sin x$.

12. $\dfrac{d \tan x}{dx} = \sec^2 x$.

13. $\dfrac{d \cot x}{dx} = -\csc^2 x$.

14. $\dfrac{d \sec x}{dx} = \sec^2 x \cdot \sin x$.

15. $\dfrac{d \csc x}{dx} = -\csc^2 x \cdot \cos x$.

16. $\dfrac{d \sin^{-1} x}{dx} = -\dfrac{d \cos^{-1} x}{dx} = \dfrac{1}{\sqrt{1 - x^2}}$.

17. $\dfrac{d \tan^{-1} x}{dx} = -\dfrac{d \cot^{-1} x}{dx} = \dfrac{1}{1 + x^2}$.

18. $\dfrac{d \sec^{-1} x}{dx} = -\dfrac{d \csc^{-1} x}{dx} = \dfrac{1}{x\sqrt{x^2 - 1}}$.

19. $\dfrac{d \sinh x}{dx} = \cosh x$.

20. $\dfrac{d \cosh x}{dx} = \sinh x$.

21. $\dfrac{d\tanh x}{dx} = \operatorname{sech}^2 x.$

22. $\dfrac{d\coth x}{dx} = -\operatorname{csch}^2 x.$

23. $\dfrac{d\operatorname{sech} x}{dx} = -\operatorname{sech} x \cdot \tanh x.$

24. $\dfrac{d\operatorname{csch} x}{dx} = -\operatorname{csch} x \cdot \coth x.$

25. $\dfrac{d\sinh^{-1} x}{dx} = \dfrac{1}{\sqrt{x^2+1}}.$

26. $\dfrac{d\cosh^{-1} x}{dx} = \dfrac{1}{\sqrt{x^2-1}}.$

27. $\dfrac{d\tanh^{-1} x}{dx} = \dfrac{d\coth^{-1} x}{dx} = \dfrac{1}{1-x^2}.$

28. $\dfrac{d\operatorname{sech}^{-1} x}{dx} = -\dfrac{1}{x\sqrt{1-x^2}}.$

29. $\dfrac{d\operatorname{csch}^{-1} x}{dx} = -\dfrac{1}{x\sqrt{1+x^2}}.$

30. $\dfrac{d\operatorname{gd} x}{dx} = \operatorname{sech} x.$

31. $\dfrac{d\operatorname{gd}^{-1} x}{dx} = \sec x.$

7.32

1. $\dfrac{d(y_1y_2y_3 \ldots . y_n)}{dx} = y_1y_2 \ldots . y_n\left(\dfrac{1}{y_1}\dfrac{dy_1}{dx} + \dfrac{1}{y_2}\dfrac{dy_2}{dx} + \ldots + \dfrac{1}{y_n}\dfrac{dy_n}{dx}\right).$

2. $\dfrac{d\left(\dfrac{u}{v}\right)}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$

3. $\dfrac{da^u}{dx} = a^u\dfrac{du}{dx}\log a.$

4. $\dfrac{de^u}{dx} = e^u\dfrac{du}{dx}.$

5. $\dfrac{df(u)}{dx} = \dfrac{df(u)}{du}\cdot\dfrac{du}{dx}.$

7.33 Derivative of a Definite Integral.

1. $\dfrac{d}{da}\displaystyle\int_{\psi(a)}^{\phi(a)} f(x, a)dx = f(\phi(a), a)\dfrac{d\phi(a)}{da} - f(\psi(a), a)\dfrac{d\psi(a)}{da} + \int_{\psi(a)}^{\phi(a)}\dfrac{d}{da}f(x, a)dx.$

2. $\dfrac{d}{da}\displaystyle\int_b^a f(x)dx = f(a).$

3. $\dfrac{d}{db}\displaystyle\int_b^a f(x)dx = -f(b).$

7.35 Higher Derivatives.

7.351 Leibnitz's Theorem. If u and v are functions of x,

$$\frac{d^n(uv)}{dx^n} = u\frac{d^nv}{dx^n} + \frac{n}{1!}\frac{du}{dx}\frac{d^{n-1}v}{dx^{n-1}} + \frac{n(n-1)}{2!}\frac{d^2u}{dx^2}\frac{d^{n-2}v}{dx^{n-2}} + \frac{n(n-1)(n-2)}{3!}\frac{d^3u}{dx^3}\frac{d^{n-3}v}{dx^{n-3}} + \ldots\ldots + v\frac{d^nu}{dx^n}.$$

7.352 Symbolically,

$$\frac{d^n(uv)}{dx^n} = (u + v)^{(n)},$$

where

$$u^0 = u, \quad v^0 = v.$$

7.353

$$\frac{d^n e^{ax}u}{dx^n} = e^{ax}\left(a + \frac{d}{dx}\right)^n u.$$

7.354 If $\phi\left(\frac{d}{dx}\right)$ is a polynomial in $\frac{d}{dx}$,

$$\phi\left(\frac{d}{dx}\right)e^{ax}u = e^{ax}\phi\left(a + \frac{d}{dx}\right)u.$$

7.355 Euler's Theorem. If u is a homogeneous function of the nth degree of r variables, $x_1, x_2, \ldots x_r$,

$$\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \ldots\ldots + x_r\frac{\partial}{\partial x_r}\right)^m u = n^m u,$$

where m may be any integer, including 0.

7.36 Derivatives of Functions of Functions.

7.361 If $f(x) = F(y)$, and $y = \phi(x)$,

1. $$\frac{d^n}{dx^n}f(x) = \frac{U_1}{1!}F'(y) + \frac{U_2}{2!}F''(y) + \frac{U_3}{3!}F'''(y) + \ldots\ldots + \frac{U_n}{n!}F^{(n)}(y),$$

where

2. $$U_k = \frac{\partial^n}{\partial x^n}y^k - \frac{k}{1!}y\frac{\partial^n}{\partial x^n}y^{k-1} + \frac{k(k-1)}{2!}y^2\frac{\partial^n}{\partial x^n}y^{k-2} - \ldots\ldots$$

7.362

1. $$(-1)^n\frac{d^n}{dx^n}F\left(\frac{1}{x}\right) = \frac{1}{x^{2n}}F^{(n)}\left(\frac{1}{x}\right) + \frac{n-1}{x^{2n-1}}\frac{n}{1!}F^{(n-1)}\left(\frac{1}{x}\right) + \frac{(n-1)(n-2)}{x^{2n-2}}\cdot\frac{n(n-1)}{2!}F^{(n-2)}\left(\frac{1}{x}\right) + \ldots\ldots$$

2. $$(-1)^n\frac{d^n}{dx^n}e^{\frac{a}{x}} = \frac{1}{x^n}e^{\frac{a}{x}}\left\{\left(\frac{a}{x}\right)^n + (n-1)\frac{n}{1!}\left(\frac{a}{x}\right)^{n-1} + (n-1)(n-2)\frac{n(n-1)}{2!}\left(\frac{a}{x}\right)^{n-2} + (n-1)(n-2)(n-3)\frac{n(n-1)(n-2)}{3!}\left(\frac{a}{x}\right)^{n-3} + \ldots\ldots\right\}.$$

7.363

1. $$\frac{d^n}{dx^n}F(x^2) = (2x)^n F^{(n)}(x^2) + \frac{n(n-1)}{1!}(2x)^{n-2}F^{(n-1)}(x^2) + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4}F^{(n-2)}(x^2) + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!}(2x)^{n-6}F^{(n-3)}(x^2) + \dots .$$

2. $$\frac{d^n}{dx^n}e^{ax^2} = (2ax)^n e^{ax^2}\left\{1 + \frac{n(n-1)}{1!(4ax^2)} + \frac{n(n-1)(n-2)(n-3)}{2!(4ax^2)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{3!(4ax^2)^3} + \dots \right\}.$$

3. $$\frac{d^n}{dx^n}(1+ax^2)^\mu = \frac{\mu(\mu-1)(\mu-2)\dots(\mu-n+1)(2ax)^n}{(1+ax^2)^{n-\mu}}\left\{1 + \frac{n(n-1)}{1\cdot(\mu-n+1)}\,\frac{(1+ax^2)}{4ax^2} + \frac{n(n-1)(n-2)(n-3)}{2!(\mu-n+1)(\mu-n+2)}\left(\frac{1+ax^2}{4ax^2}\right)^2 + \dots \right\}.$$

4. $$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{m-\frac{1}{2}} = (-1)^{m-1}\frac{1\cdot3\cdot5\dots(2m-1)}{m}\sin(m\cos^{-1}x).$$

7.364

1. $$\frac{d^n}{dx^n}F(\sqrt{x}) = \frac{F^{(n)}(\sqrt{x})}{(2\sqrt{x})^n} - \frac{n(n-1)}{1!}\,\frac{F^{(n-1)}(\sqrt{x})}{(2\sqrt{x})^{n+1}} + \frac{(n+1)n(n-1)(n-2)}{2!}\,\frac{F^{(n-2)}(\sqrt{x})}{(2\sqrt{x})^{n+2}} - \dots .$$

2. $$\frac{d^n}{dx^n}(1+a\sqrt{x})^{2n-1} = \frac{1\cdot3\cdot5\dots(2n-1)}{2^n}\,\frac{a}{\sqrt{x}}\left(a^2-\frac{1}{x}\right)^{n-1}.$$

7.365

1. $$\frac{d^n}{dx^n}F(e^x) = \frac{E_1}{1!}e^xF'(e^x) + \frac{E_2}{2!}e^{2x}F''(e^x) + \frac{E_3}{3!}e^{3x}F'''(e^x) + \dots$$

where

2. $$E_k = k^n - \frac{k}{1!}(k-1)^n + \frac{k(k-1)}{2!}(k-2)^n - \dots .$$

3. $$\frac{d^n}{dx^n}\frac{1}{1+e^{2x}} = -E_1e^x\frac{\sin(2\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^2}} + E_2e^{2x}\frac{\sin(3\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^3}} - E_3e^{3x}\frac{\sin(4\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^4}} + \dots .$$

4. $$\frac{d^n}{dx^n}\frac{e^x}{1+e^{2x}} = -E_1e^x\frac{\cos(2\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^2}} + E_2e^{2x}\frac{\cos(3\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^3}} - E_3e^{3x}\frac{\cos(4\tan^{-1}e^{-x})}{\sqrt{(1+e^{2x})^4}} + \dots .$$

7.366

1. $$\frac{d^n}{dx^n}F(\log x) = \frac{1}{x^n}\left\{\overset{n}{C}_0 F^{(n)}(\log x) - \overset{n}{C}_1 F^{(n-1)}(\log x) + \overset{n}{C}_2 F^{(n-2)}(\log x) - \dots\right\}.$$

$\overset{n}{C}_0 = 1,$

$\overset{n}{C}_1 = 1 + 2 + 3 + \dots + (n-1) \qquad = \dfrac{n(n-1)}{2},$

$$\begin{aligned}\overset{n}{C}_2 = 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + &\dots + 1\cdot(n-1)\\ + 2\cdot 3 + 2\cdot 4 + &\dots + 2\cdot(n-1)\\ + 3\cdot 4 + &\dots + 3\cdot(n-1)\\ + &\dots\dots\\ &+ (n-2)(n-1) = \frac{n(n-1)(n-2)(3n-1)}{24}.\end{aligned}$$

2. $\overset{n+1}{C}_k = \overset{n}{C}_k + n\overset{n}{C}_{k-1}.$

3. $\overset{-n}{C}_k = \overset{-(n-1)}{C}_k + n\overset{-n}{C}_{k-1}.$

$\overset{n}{C}_0 = 1 \quad \overset{k}{C}_k = 0, \qquad\qquad \overset{-n}{C}_0 = 1 \quad \overset{-1}{C}_k = 1,$

$\overset{2}{C}_1 = 1 \quad \overset{3}{C}_1 = 3 \quad \overset{4}{C}_1 = 6, \qquad \overset{-2}{C}_1 = 3 \quad \overset{-3}{C}_1 = 6 \quad \overset{-4}{C}_1 = 10,$

$\overset{3}{C}_2 = 2 \quad \overset{4}{C}_2 = 11, \qquad \overset{-2}{C}_2 = 7 \quad \overset{-3}{C}_2 = 25 \quad \overset{-4}{C}_2 = 65,$

$\overset{4}{C}_3 = 6. \qquad \overset{-2}{C}_3 = 15 \quad \overset{-3}{C}_3 = 90 \quad \overset{-4}{C}_3 = 350.$

7.367 Table of $\overset{n}{C}_k$.

$n =$	-4	-3	-2	-1	$+1$	$+2$	$+3$	$+4$	$+5$	$+6$	$+7$	$+8$	$+9$
$C_0 =$	1	1	1	1	1	1	1	1	1	1	1	1	1
$C_1 =$	10	6	3	1	...	1	3	6	10	15	21	28	36
$C_2 =$	65	25	7	1	...	...	2	11	35	85	175	322	546
$C_3 =$	350	90	15	1	...	...	...	6	50	225	735	1960	4536
$C_4 =$	1701	301	31	1	...	...	...	...	24	274	1624	6769	22449
$C_5 =$	7770	966	63	1	...	...	...	...	...	120	1764	13132	67284
$C_6 =$	34105	3025	127	1	...	...	...	...	...	...	720	13068	118124
$C_7 =$	145750	9330	225	1	...	...	...	...	...	...	...	5040	109584
$C_8 =$	611501	28501	511	1	...	...	...	...	...	...	...	...	40320

7.368

1. $$\frac{d^n}{dx^n}(\log x)^p = \frac{(-1)^{n-1}}{x^n}\left\{ \overset{n}{C}_{n-1}p(\log x)^{p-1} - \overset{n}{C}_{n-2}p(p-1)(\log x)^{p-2} + \overset{n}{C}_{n-3}p(p-1)(p-2)(\log x)^{p-3} - \ldots \right\},$$

where p is a positive integer. If $n<p$ there are n terms in the series. If $n \geqslant p$,

2. $$\frac{d^n}{dx^n}(\log x)^p = \frac{(-1)^{n-1}}{x^n}\left\{ \overset{n}{C}_{n-1}p(\log x)^{p-1} - \overset{n}{C}_{n-2}p(p-1)(\log x)^{p-2} + \ldots\ldots + (-1)^{p+1}\overset{n}{C}_{n-p}p(p-1)(p-2)\ldots\ldots 2\cdot 1 \right\}.$$

7.369 $$\left\{\log(1+x)\right\}^p = \overset{p}{C}_0 x^p - \overset{p+1}{C_1}\frac{x^{p+1}}{p+1} + \overset{p+2}{C_2}\frac{x^{p+2}}{(p+1)(p+2)} - \ldots\ldots$$
$$-1 < x < +1.$$

7.37 Derivatives of Powers of Functions. If $y = \phi(x)$.

1. $$\frac{d^n}{dx^n}y^p = p\binom{n-p}{n}\left\{-\binom{n}{1}\frac{1}{p-1}y^{p-1}\frac{d^n y}{dx^n} + \binom{n}{2}\frac{1}{p-2}y^{p-2}\frac{d^n y^2}{dx^n} - \ldots\ldots\right\}.$$

2. $$\frac{d^n}{dx^n}\log y = \binom{n}{1}\frac{1}{1\cdot y}\frac{d^n y}{dx^n} - \binom{n}{2}\frac{1}{2\cdot y^2}\frac{d^n y^2}{dx^n} + \binom{n}{3}\frac{1}{3\cdot y^3}\frac{d^n y^3}{dx^n} - \ldots\ldots$$

7.38

1. $$\frac{d^n(a+bx)^m}{dx^n} = m(m-1)(m-2)\ldots\ldots(m-[n-1])\,b^n(a+bx)^{m-n}.$$

2. $$\frac{d^n(a+bx)^{-1}}{dx^n} = (-1)^n\frac{n!\,b^n}{(a+bx)^{n+1}}.$$

3. $$\frac{d^n(a+bx)^{-\frac{1}{2}}}{dx^n} = (-1)^n\frac{1\cdot 3\cdot 5\ldots\ldots(2n-1)}{2^n(a+bx)^{n+\frac{1}{2}}}b^n.$$

4. $$\frac{d^n\log(a+bx)}{dx^n} = (-1)^{n-1}\frac{(n-1)!\,b^n}{(a+bx)^n}.$$

5. $$\frac{d^n e^{ax}}{dx^n} = a^n e^{ax}.$$

6. $$\frac{d^n\sin x}{dx^n} = \sin(\tfrac{1}{2}n\pi + x).$$

7. $$\frac{d^n\cos x}{dx^n} = \cos(\tfrac{1}{2}n\pi + x).$$

8. $\dfrac{d^n}{dx^n}\left(\dfrac{\log x}{x}\right) = (-1)^n \dfrac{n!}{x^{n+1}} \left\{ \log x - \left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n}\right)\right\}.$

9. $$\frac{d^{n+1}}{dx^{n+1}} \sin^{-1}x = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{2^n(1-x)^n\sqrt{1-x^2}}\left\{1 - \frac{1}{2n-1}\binom{n}{1}\frac{1-x}{1+x}\right.$$
$$+ \frac{1\cdot 3}{(2n-1)(2n-3)}\binom{n}{2}\left(\frac{1-x}{1+x}\right)^2 - \frac{1\cdot 3\cdot 5}{(2n-1)(2n-3)(2n-5)}\binom{n}{3}\left(\frac{1-x}{1+x}\right)^3$$
$$\left. + \ldots \right\}.$$

10. $\dfrac{d^n}{dx^n}(\tan^{-1}x) = (-1)^{n-1}\dfrac{(n-1)!}{(1+x^2)^{\frac{n}{2}}}\sin\left(n\tan^{-1}\dfrac{1}{x}\right).$

7.39 Derivatives of Implicit Functions.

7.391 If y is a function of x, and $f(x, y) = 0$.

1. $$\frac{dy}{dx} = -\frac{\dfrac{\partial}{\partial x}}{\dfrac{\partial f}{\partial y}}.$$

2. $$\frac{d^2y}{dx^2} = -\frac{\left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial x\partial y} + \left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial y^2}}{\left(\dfrac{\partial f}{\partial y}\right)^3}$$

7.392 If z is a function of x and y, and $f(x, y, z) = 0$.

1. $$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial z}}; \quad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial z}}.$$

2. $$\frac{\partial^2 z}{\partial x^2} = -\frac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial x}\dfrac{d^2 f}{\partial x\partial z} + \left(\dfrac{\partial f}{\partial x}\right)^2\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$$

3. $$\frac{\partial^2 z}{\partial y^2} = -\frac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial y^2} - 2\dfrac{\partial f}{\partial z}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial y\partial z} + \left(\dfrac{\partial f}{\partial y}\right)^2\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$$

4. $$\frac{\partial^2 z}{\partial x\partial y} = -\frac{\left(\dfrac{\partial f}{\partial z}\right)^2\dfrac{\partial^2 f}{\partial x\partial y} - \dfrac{\partial f}{\partial z}\left(\dfrac{\partial f}{\partial x}\dfrac{\partial^2 f}{\partial y\partial z} + \dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial x\partial z}\right) + \dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}\dfrac{\partial^2 f}{\partial z^2}}{\left(\dfrac{\partial f}{\partial z}\right)^3}.$$

VIII. DIFFERENTIAL EQUATIONS.

8.000 Ordinary differential equations of the first order. General form:

$$\frac{dy}{dx} = f(x, y).$$

8.001 Variables are separable. $f(x, y)$ is of, or can be reduced to, the form:

$$f(x, y) = -\frac{X}{Y},$$

where X is a function of x alone and Y is a function of y alone.
The solution is:

$$\int X\,dx + \int Y\,dy = C.$$

8.002 Linear equations of the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Solution:

$$y = e^{-\int P(x)dx}\left\{\int Q(x)e^{\int P(x)dx}dx + C\right\}.$$

8.003 Equations of the form:

$$\frac{dy}{dx} + P(x)y = y^n Q(x).$$

Solution:

$$\frac{1}{y^{n-1}}e^{-(n-1)\int P(x)dx} + (n-1)\int Q(x)e^{-(n-1)\int P(x)dx}dx = C.$$

8.010 Homogeneous equations of the form:

$$\frac{dy}{dx} = -\frac{P(x, y)}{Q(x, y)},$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous functions of x and y of the same degree. The change of variable:

$$y = vx,$$

gives the solution:

$$\int \frac{dv}{\dfrac{P(1, v)}{Q(1, v)} + v} + \log x = C.$$

8.011 Equations of the form:

$$\frac{dy}{dx} = \frac{a'x + b'y + c'}{ax + by + c}.$$

If $ab' - a'b \neq 0$, the substitution

$$x = x' + p, \quad y = y' + q,$$

where

$$ap + bq + c = 0,$$
$$a'p + b'q + c' = 0,$$

renders the equation homogeneous, and it may be solved by **8.010.**

If $ab' - a'b = 0$ and $b' \neq 0$, the change of variables to either x and z or y and z by means of

$$z = ax + by,$$

will make the variables separable **(8.001).**

8.020 Exact differential equations. The equation,

$$P(x, y)dx + Q(x, y)dy = 0,$$

is exact if,

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}.$$

The solution is:

$$\int P(x, y)dx + \int \left\{ Q(x, y) - \frac{\partial}{\partial y}\int P(x, y)dx \right\} dy = C,$$

or

$$\int Q(x, y)dy + \int \left\{ P(x, y) - \frac{\partial}{\partial x}\int Q(x, y)dy \right\} dx = C.$$

8.030 Integrating factors. $v(x, y)$ is an integrating factor of

$$P(x, y)\ dx + Q(x, y)\ dy = 0,$$

if

$$\frac{\partial}{\partial x}(vQ) = \frac{\partial}{\partial y}(vP).$$

8.031 If one only of the functions $Px + Qy$ and $Px - Qy$ is equal to 0, the reciprocal of the other is an integrating factor of the differential equation.

8.032 Homogeneous equations. If neither $Px + Qy$ nor $Px - Qy$ is equal to 0

$\dfrac{1}{Px + Qy}$ is an integrating factor of the equation if it is homogeneous.

8.033 An equation of the form,

$$P(x, y)y\,dx + Q(x, y)x\,dy = 0,$$

has an integrating factor:

$$\frac{1}{xP - yQ}.$$

8.034 If

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Q} = F(x)$$

is a function of x only, an integrating factor is

$$e^{\int F(x)dx}.$$

8.035 If

$$\frac{\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}}{P} = F(y)$$

is a function of y only, an integrating factor is

$$e^{\int F(y)dy}.$$

8.036 If

$$\frac{\dfrac{\partial P}{\partial y} - \dfrac{\partial Q}{\partial x}}{Qy - Px} = F(xy)$$

is a function of the product xy only, an integrating factor is

$$e^{\int F(xy)d(xy)}.$$

8.037 If

$$\frac{x^2\left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)}{Px + Qy} = F\left(\frac{y}{x}\right)$$

is a function of the quotient $\frac{y}{x}$ only, an integrating factor is

$$e^{\int F\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)}.$$

8.040 Ordinary differential equations of the first order and of degree higher than the first.

Write:

$$\frac{dy}{dx} = p.$$

General form of equation:

$$f(x, y, p) = 0.$$

8.041 The equation can be solved as an algebraic equation in p. It can be written

$$(p - R_1)(p - R_2) \ldots\ldots\ldots (p - R_n) = 0.$$

The differential equations:

$$p = R_1(x, y),$$
$$p = R_2(x, y),$$
$$\ldots\ldots$$

may be solved by the previous methods. Write the solutions:

$$f_1(x, y, c) = 0; \quad f_2(x, y, c) = 0; \quad \ldots\ldots\ldots$$

where c is the same arbitrary constant in each. The solution of the given differential equation is:

$$f_1(x, y, c) f_2(x, y, c) \ldots\ldots\ldots\ldots f_n(x, y, c) = 0.$$

8.042 The equation can be solved for y:

1. $$y = f(x, p).$$

Differentiate with respect to x:

2. $$p = \psi\left(x, p, \frac{dp}{dx}\right).$$

It may be possible to integrate (2) regarded as an equation in the two variables x, p, giving a solution

3. $$\phi(x, p, c) = 0.$$

If p is eliminated between (1) and (3) the result will be the solution of the given equation.

8.043 The equation can be solved for x:

1. $$x = f(y, p).$$

Differentiate with respect to y:

2. $$\frac{1}{p} = \psi\left(y, p, \frac{dp}{dy}\right).$$

If a solution of (2) can be found:

3. $$\phi(y, p, c) = 0.$$

Eliminate p between (1) and (3) and the result will be the solution of the giv equation.

8.044 The equation does not contain x:

$$f(y, p) = 0.$$

It may be solved for p, giving,

$$\frac{dy}{dx} = F(y),$$

which can be integrated.

8.045 The equation does not contain y:

$$f(x, p) = 0.$$

It may be solved for p, giving,

$$\frac{dy}{dx} = F(x),$$

which can be integrated.
It may be solved for x, giving,

$$x = F(p),$$

which may be solved by **8.043**.

8.050 Equations homogeneous in x and y.

General form:

$$F\left(p, \frac{y}{x}\right) = 0.$$

(a) Solve for p and proceed as in **8.001**

(b) Solve for $\frac{y}{x}$:

$$y = xf(p).$$

Differentiate with respect to x:

$$\frac{dx}{x} = \frac{f'(p)dp}{p - f(p)},$$

which may be integrated.

8.060 Clairaut's differential equation:

1. $$y = px + f(p),$$

the solution is:

$$y = cx + f(c).$$

The singular solution is obtained by eliminating p between (1) and

2. $$x + f'(p) = 0.$$

8.061 The equation

1. $$y = xf(p) + \phi(p).$$

The solution is that of the linear equation of the first order:

2. $$\frac{dx}{dp} - \frac{f'(p)}{p - f(p)}x = \frac{\phi'(p)}{p - f(p)},$$

which may be solved by **8.002**. Eliminating p between (1) and the solution of (2) gives the solution of the given equation.

8.062 The equation:

$$x\phi(p) + y\psi(p) = \chi(p),$$

may be reduced to **8.061** by dividing by $\psi(p)$.

DIFFERENTIAL EQUATIONS OF AN ORDER HIGHER THAN THE FIRST

8.100 Linear equations with constant coefficients. General form:

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_n y = V(x).$$

The complete solution consists of the sum of

(a) The complementary function, obtained by solving the equation with $V(x) = 0$, and containing n arbitrary constants, and

(b) The particular integral, with no arbitrary constants.

8.101 The complementary function. Assume $y = e^{\lambda x}$. The equation for determining λ is:

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots\ldots + a_n = 0.$$

8.102 If the roots of **8.101** are all real and distinct the complementary function is:

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \ldots\ldots + c_n e^{\lambda_n x}.$$

8.103 For a pair of complex roots:

$$\mu \pm i\nu,$$

the corresponding terms in the complementary function are:

$$e^{\mu x}(A \cos \nu x + B \cos \nu x) = Ce^{\mu x} \cos (\nu x - \theta) = Ce^{\mu x} \sin (\nu x + \theta),$$

where

$$C = \sqrt{A^2 + B^2}, \quad \tan \theta = \frac{B}{A}.$$

8.104 If there are r equal real roots the terms in the complementary function corresponding to them are:

$$e^{\lambda x}(A_1 + A_2 x + A_3 x^2 + \ldots\ldots + A_r x^{r-1}),$$

where λ is the repeated root, and $A_1, A_2, \ldots\ldots, A_r$ are the r arbitrary constants.

8.105 If there are m equal pairs of complex roots the terms in the complementary function corresponding to them are:

$$\begin{aligned} &e^{\mu x}\{(A_1 + A_2 x + A_3 x^2 + \ldots\ldots + A_m x^{m-1}) \cos \nu x \\ &\qquad\qquad + (B_1 + B_2 x + B_3 x^2 + \ldots\ldots + B_m x^{m-1}) \sin \nu x\} \\ &= e^{\mu x}\{C_1 \cos (\nu x - \theta_1) + C_2 x \cos (\nu x - \theta_2) + \ldots\ldots + C_m x^{m-1} \cos (\nu x - \theta_m)\} \\ &= e^{\mu x}\{C_1 \sin (\nu x + \theta_1) + C_2 x \sin (\nu x + \theta_2) + \ldots\ldots + C_m x^{m-1} \sin (\nu x + \theta_m)\} \end{aligned}$$

where $\lambda \pm i\mu$ is the repeated root and

$$C_k = \sqrt{A_k^2 + B_k^2},$$

$$\tan \theta_k = \frac{B_k}{A_k}.$$

The particular integral.

8.110 The operator D stands for $\frac{\partial}{\partial x}$, D^2 for $\frac{\partial^2}{\partial x^2}$, ...

The differential equation **8.100** may be written:

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = f(D)y = V(x)$$

$$y = \frac{V(x)}{f(D)},$$

$$f(D) = (D - \lambda_1)(D - \lambda_2) \ldots\ldots (D - \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are determined as in **8.101**. The particular integral is:

$$y = e^{\lambda_1 x}\int e^{(\lambda_2-\lambda_1)x}dx\int e^{(\lambda_3-\lambda_2)x}dx \ldots\ldots \int e^{-\lambda_n(x)}V(x)dx.$$

8.111 $\frac{1}{f(D)}$ may be resolved into partial fractions:

$$\frac{1}{f(D)} = \frac{N_1}{D-\lambda_1} + \frac{N_2}{D-\lambda_2} + \ldots\ldots + \frac{N_n}{D-\lambda_n}.$$

The particular integral is:

$$y = N_1 e^{\lambda_1 x}\int e^{-\lambda_1 x}V(x)dx + N_2 e^{\lambda_2 x}\int e^{-\lambda_2 x}V(x)dx + \ldots\ldots$$
$$+ N_n e^{\lambda_n x}\int e^{-\lambda_n x}V(x)dx.$$

THE PARTICULAR INTEGRAL IN SPECIAL CASES

8.120 $V(x) = \text{const.} = c$,

$$y = \frac{c}{a_n}.$$

8.121 $V(x)$ is a rational integral function of x of the mth degree. Expand $\frac{1}{f(D)}$ in ascending powers of D, ending with D^m. Apply the operators D, D^2, ..., D^m to each term of $V(x)$ separately and the particular integral will be the sum of the results of these operations.

8.122
$$V(x) = ce^{kx},$$
$$y = \frac{c}{f(k)}\, e^{kx},$$
unless k is a root of $f(D) = 0$. If k is a multiple root of order r of $f(D) = 0$
$$y = \frac{cx^r e^{kx}}{r!\psi(k)},$$
where
$$f(D) = (D - k)^r \psi(D).$$

8.123
$$V(x) = c \cos(kx + \alpha).$$
If ik is not a root of $f(D) = 0$ the particular integral is the real part of
$$\frac{c}{f(ik)}\, e^{i(kx+\alpha)}.$$
If ik is a multiple root of order r of $f(D) = 0$ the particular integral is the real part of
$$\frac{cx^r e^{i(kx+\alpha)}}{f^{(r)}(ik)},$$
where $f^{(r)}(ik)$ is obtained by taking the rth derivative of $f(D)$ with respect to D, and substituting ik for D.

8.124
$$V(x) = c \sin(kx + \alpha).$$
If ik is not a root of $f(D) = 0$ the particular integral is the real part of
$$\frac{-ic\, e^{i(kx+\alpha)}}{f(ik)}.$$
If ik is a multiple root of order r of $f(D) = 0$ the particular integral is the real part of
$$\frac{-icx^r e^{i(kx+\alpha)}}{f^{(r)}(ik)}.$$

8.125
$$V(x) = ce^{kx} \cdot X,$$
where X is any function of x.
$$y = ce^{kx} \frac{1}{f(D + k)} X.$$
If X is a rational integral function of x this may be evaluated by the method of **8.121**.

8.126
$$V(x) = c \cos(kx + \alpha) \cdot X,$$
where X is any function of x. The particular integral is the real part of
$$ce^{i(kx+\alpha)} \frac{1}{f(D + ik)} X.$$

8.127
$$V(x) = c \sin(kx + \alpha) \cdot X.$$
The particular integral is the real part of
$$-ice^{i(kx+\alpha)} \frac{1}{f(D + ik)} X.$$

8.128 $$V(x) = ce^{\beta x}\cos(kx + \alpha).$$

If $(\beta + ik)$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$ce^{i(kx+\alpha)}\frac{1}{f(\beta + ik)}e^{\beta x}.$$

If $(\beta + ik)$ is a multiple root of order r of $f(D) = 0$ the particular integral is the real part of

$$\frac{ce^{i(kx+\alpha)}x^r e^{\beta x}}{f^{(r)}(\beta + ik)},$$

where $f^{(r)}(\beta + ik)$ is formed as in **8.123.**

8.129 $$V = ce^{\beta x}\sin(kx + \alpha).$$

If $(\beta + ik)$ is not a root of $f(D) = 0$ the particular integral is the real part of

$$\frac{-ice^{i(kx+\alpha)}e^{\beta x}}{f(\beta + ik)}.$$

If $(\beta + ik)$ is a multiple root of order r of $f(D) = 0$ the particular integral is the real part of

$$\frac{-ice^{i(kx+\alpha)}x^r e^{\beta x}}{f^{(r)}(\beta + ik)}.$$

8.130 $$V(x) = x^m X,$$

where X is any function of x.

$$y = x^m\frac{1}{f(D)}X + mx^{m-1}\left\{\frac{d}{dD}\frac{1}{f(D)}\right\}X + \frac{m(m-1)}{2!}x^{m-2}\left\{\frac{d^2}{dD^2}\frac{1}{f(D)}\right\}X + \ldots\ldots$$

The series must be extended to the $(m+1)$th term.

8.200 Homogeneous linear equations. General form:

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}x\frac{dy}{dx} + a_ny = V(x).$$

Denote the operator:

$$x\frac{d}{dx} = \theta,$$

$$x^m\frac{d^m}{dx^m} = \theta(\theta - 1)(\theta - 2)\ \ldots\ldots\ (\theta - m + 1).$$

The differential equation may be written:

$$F(\theta)\cdot y = V(x).$$

The complete solution is the sum of the complementary function, obtained by solving the equation with $V(x) = 0$, and the particular integral.

8.201 The complementary function.

$$y = c_1x^{\lambda_1} + c_2x^{\lambda_2} + \ldots\ldots + c_nx^{\lambda_n},$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the n roots of

$$F(\lambda) = 0$$

if the roots are all distinct.

If λ_k is a multiple root of order r, the corresponding terms in the complementary function are:

$$x^{\lambda_k}\{b_1 + b_2 \log x + b_3 (\log x)^2 + \ldots\ldots + b_r (\log x)^{r-1}\}.$$

If $\lambda = \mu \pm i\nu$ is a pair of complex roots, of order r, the corresponding terms in the complementary function are:

$$x^{\mu}\{[A_1 + A_2 \log x + A_3 (\log x)^2 + \ldots\ldots + A_r (\log x)^{r-1}] \cos (\nu \log x)$$
$$+ [B_1 + B_2 \log x + B_3 (\log x)^2 + \ldots\ldots + B_r(\log x)^{r-1}] \sin (\nu \log x)\}.$$

8.202 The particular integral.

If

$$F(\theta) = (\theta - \lambda_1)(\theta - \lambda_2) \ldots\ldots (\theta - \lambda_n),$$

$$y = x^{\lambda_1}\int x^{\lambda_2-\lambda_1-1}dx \int x^{\lambda_3-\lambda_2-1}dx \ldots\ldots \int x^{\lambda_n-\lambda_{n-1}-1}V(x)dx.$$

8.203 The operator $\frac{1}{F(\theta)}$ may be resolved into partial fractions:

$$\frac{1}{F(\theta)} = \frac{N_1}{\theta - \lambda_1} + \frac{N_2}{\theta - \lambda_2} + \ldots\ldots + \frac{N_n}{\theta - \lambda_n},$$

$$y = N_1x^{\lambda_1}\int x^{-\lambda_1-1}V(x)dx + N_2x^{\lambda_2}\int x^{-\lambda_2-1}V(x)dx$$
$$+ \ldots\ldots + N_nx^{\lambda_n}\int x^{-\lambda_n-1}V(x)dx.$$

The particular integral in special cases.

8.210

$$V(x) = cx^k,$$

$$y = \frac{c}{F(k)}x^k,$$

unless k is a root of $F(\theta) = 0$.

If k is a multiple root of order r of $F(\theta) = 0$.

$$y = \frac{c (\log x)^r}{F^{(r)}(k)},$$

where $F^{(r)}(k)$ is obtained by taking the rth derivative of $F(\theta)$ with respect to θ and after differentiation substituting k for θ.

8.211
$$V(x) = cx^k X,$$
where X is any function of x.
$$y = cx^k \frac{1}{F(0+k)} X.$$

8.220 The differential equation:

$$(a+bx)^n \frac{d^n y}{dx^n} + (a+bx)^{n-1} a_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + (a+bx)a_{n-1}\frac{dy}{dx} + a_n y = V(x),$$

may be reduced to the homogeneous linear equation (**8.200**) by the change of variable

$$z = a + bx.$$

It may be reduced to a linear equation with constant coefficients by the change of variable:

$$e^z = a + bx.$$

8.230 The general linear equation. General form:

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + P_{n-1}\frac{dy}{dx} + P_n = V,$$

where $P_0, P_1, \ldots\ldots, P_n, V$ are functions of x only.

The complete solution is the sum of:

(a) The complementary function, which is the general solution of the equation with $V = 0$, and containing n arbitrary constants, and

(b) The particular integral.

8.231 Complementary Function. If $y_1, y_2, \ldots\ldots, y_n$ are n independent solutions of **8.230** with $V = 0$, the complementary function is

$$y = c_1 y_1 + c_2 y_2 + \cdots\cdots + c_n y_n.$$

The conditions that $y_1, y_2, \ldots\ldots, y_n$ be n independent solutions is that the determinant $\Delta \neq 0$.

$$\Delta = \begin{vmatrix} \frac{d^{n-1}y_1}{dx^{n-1}} & \frac{d^{n-1}y_2}{dx^{n-1}} & \cdots\cdots & \frac{d^{n-1}y_n}{dx^{n-1}} \\ \frac{d^{n-2}y_1}{dx^{n-2}} & \frac{d^{n-2}y_2}{dx^{n-2}} & \cdots\cdots & \frac{d^{n-2}y_n}{dx^{n-2}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \cdots\cdots & \frac{dy_n}{dx} \\ y_1 & y_2 & \cdots\cdots & y_n \end{vmatrix}$$

When $\Delta \neq 0$:

$$\Delta = Ce^{-\int \frac{P_1}{P_0} dx}.$$

8.232 The particular integral. If Δ_k is the minor of $\frac{d^{n-1}y_k}{dx^{n-1}}$ in Δ, the particular integral is:

$$y = y_1\int\frac{V\Delta_1}{P_0\Delta}dx + y_2\int\frac{V\Delta_2}{P_0\Delta}dx + \ldots\ldots + y_n\int\frac{V\Delta_n}{P_0\Delta}dx.$$

8.233 If y_1 is one integral of the equation **8.230** with $v = 0$, the substitution

$$y = uy_1, \quad v = \frac{du}{dx},$$

will result in a linear equation of order $n - 1$.

8.234 If $y_1, y_2, \ldots\ldots, y_{n-1}$ are $n - 1$ independent integrals of **8.230** with $V = 0$ the complete solution is:

$$y = \sum_{k=1}^{n-1} y\,c_{kk} + c_n\sum_{k=1}^{n-1} y_k\int\frac{\Delta_k}{\Delta^2}e^{-\int\frac{P_1}{P_0}dx}dx$$

where Δ is the determinant:

$$\Delta = \begin{vmatrix} \frac{d^{n-2}y_1}{dx^{n-2}} & \frac{d^{n-2}y_2}{dx^{n-2}} & \ldots\ldots & \frac{d^{n-2}y_{n-1}}{dx^{n-2}} \\ \frac{d^{n-3}y_1}{dx^{n-3}} & \frac{d^{n-3}y_2}{dx^{n-3}} & \ldots\ldots & \frac{d^{n-3}y_{n-1}}{dx^{n-3}} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \ldots\ldots & \frac{dy_{n-1}}{dx} \\ y_1 & y_2 & \ldots\ldots & y_{n-1} \end{vmatrix}$$

and Δ_k is the minor of $\frac{d^{n-2}y_k}{dx^{n-2}}$ in Δ.

SYMBOLIC METHODS

8.240 Denote the operators:

$$\frac{d}{dx} = D$$

$$x\frac{d}{dx} = \theta.$$

8.241 If X is a function of x:

1. $$(D - m)^{-1}X = e^{mx}\int e^{-mx}Xdx.$$

2. $$(D - m)^{-1}0 = ce^{mx}.$$

3. $$(\theta - m)^{-1}X = x^m\int x^{-m-1}Xdx.$$

4. $$(\theta - m)^{-1}0 = cx^m.$$

8.242 If $F(D)$ is a polynomial in D,

1. $$F(D)e^{mx} = e^{mx}F(m).$$

2. $$F(D)e^{mx}X = e^{mx}F(D+m)X.$$

3. $$e^{mx}F(D)X = F(D-m)e^{mx}X.$$

8.243 If $F(\theta)$ is a polynomial in θ,

1. $$F(\theta)x^m = x^mF(m).$$

2. $$F(\theta)x^mX = x^mF(\theta+m)X.$$

3. $$x^mF(\theta)X = F(\theta-m)x^mX.$$

8.244 $$x^m\frac{d^m}{dx^m} = \theta(\theta-1)(\theta-2)\ldots\ldots(\theta-m+1).$$

INTEGRATION IN SERIES

8.250 If a linear differential equation can be expressed in the symbolic form:

$$[x^mF(\theta)+f(\theta)]y=0,$$

where $F(\theta)$ and $f(\theta)$ are polynomials in θ, the substitution,

$$y=\sum_{n=0}^{\infty}a_nx^{\rho+nm},$$

leads to the equations,

$$\begin{aligned} a_0f(\rho)&=0,\\ a_0F(\rho)+a_1f(\rho+m)&=0,\\ a_1F(\rho+m)+a_2f(\rho+2m)&=0,\\ a_2F(\rho+2m)+a_3f(\rho+3m)&=0.\\ \ldots\ldots&\\ \ldots\ldots& \end{aligned}$$

8.251 The equation

$$f(\rho)=0,$$

is the "indicial equation." If it is satisfied a_0 may be chosen arbitrarily, and the other coefficients are then determined.

8.252 An equation:

$$\left[F(\theta)+\phi(\theta)\frac{d^m}{dx^m}\right]y=0,$$

may be reduced to the form **8.250**, where,

$$f(\theta)=\phi(\theta-m)\,\theta(\theta-1)(\theta-2)\ldots\ldots(\theta-m+1).$$

If the degree of the polynomial f is greater than that of F the series always converges; if the degree of f is less than that of F the series always diverges.

ORDINARY DIFFERENTIAL EQUATIONS OF SPECIAL TYPES

8.300

$$\frac{d^n y}{dx^n} = X,$$

where X is a function of x only.

$$y = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}\, T\,dt + c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1}x + c_n,$$

where T is the same function of t that X is of x.

8.301

$$\frac{d^2 y}{dx^2} = Y,$$

where Y is a function of y only.
If

$$\psi(y) = 2\int Y\,dy,$$

the solution is:

$$\int \frac{dy}{\{\psi(y) + c_1\}^{\frac{1}{2}}} = x + c_2.$$

8.302

$$\frac{d^n y}{dx^n} = F\left(\frac{d^{n-1}y}{dx^{n-1}}\right).$$

Put

$$\frac{d^{n-1}y}{dx^{n-1}} = Y; \quad \frac{dY}{dx} = F(Y),$$

$$x + c_1 = \int \frac{dY}{F(Y)} = \psi(Y),$$

$$Y = \phi(x + c_1),$$

$$\frac{d^{n-1}y}{dx^{n-1}} = \phi(x + c_1),$$

and this equation may be solved by **8.300**.

Or the equation can be solved:

$$y = \int \frac{dY}{F(Y)} \int \frac{dY}{F(Y)} \cdot \cdot \cdot \cdot \cdot \cdot \int \frac{Y\,dY}{F(Y)},$$

where the integration is to be carried out from right to left and an arbitrary constant added after each integration. Eliminating Y between this result and

$$Y = \phi(x + c_1)$$

gives the solution.

8.303

$$\frac{d^n y}{dx^n} = F\left(\frac{d^{n-2}y}{dx^{n-2}}\right).$$

Put

$$\frac{d^{n-2}y}{dx^{n-2}} = Y,$$

$$\frac{d^2Y}{dx^2} = F(Y),$$

which may be solved by **8.301**. If the solution can be expressed:

$$Y = \phi(x),$$

$n - 2$ integrations will solve the given differential equation.

Or putting

$$\psi(y) = 2\int^{\bullet} Y\,dy,$$

$$y = \int \frac{dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}}\int \frac{dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}} \cdots\cdots \int \frac{Y\,dY}{\{c_1 + \psi(Y)\}^{\frac{1}{2}}},$$

where the integration is to be carried out from right to left and an arbitrary constant added after each integration. The solution of the given differential equation is obtained by elimination between this result and

$$Y = \phi(x).$$

8.304 Differential equations of the second order in which the independent variable does not appear. General type:

$$F\left(y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

Put

$$p = \frac{dy}{dx}, \quad p\frac{dp}{dy} = \frac{d^2y}{dx^2}.$$

A differential equation of the first order results:

$$F\left(y, p, p\frac{dp}{dy}\right) = 0.$$

If the solution of this equation is:

$$p = f(y),$$

the solution of the given equation is,

$$x + c_2 = \int^{\bullet} \frac{dy}{f(y)}.$$

8.305 Differential equations of the second order in which the dependent variable does not appear. General type:

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0.$$

Put

$$p = \frac{dy}{dx}, \quad \frac{dp}{dx} = \frac{d^2y}{dx^2}.$$

A differential equation of the first order results:

$$F\left(x, p, \frac{dp}{dx}\right) = 0.$$

If the solution of this equation is:

$$p = f(x),$$

the solution of the given equation is:

$$y = c_2 + \int f(x)dx.$$

8.306 Equations of an order higher than the second in which either the independent or the dependent variable does not appear. The substitution:

$$\frac{dy}{dx} = p,$$

as in **8.304** and **8.305** will result in an equation of an order less by unity than the given equation.

8.307 Homogeneous differential equations. If y is assumed to be of dimensions n, x of dimensions 1, $\frac{dy}{dx}$ of dimensions $(n-1)$, $\frac{d^2y}{dx^2}$ of dimensions $(n-2)$, ... then if every term has the same dimensions the equation is homogeneous. If the independent variable is changed to θ and the dependent variable changed to z by the relations,

$$x = e^\theta, \quad y = ze^{n\theta},$$

the resulting equation will be one in which the independent variable does not appear and its order can be lowered by unity by **8.306**.

If $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ are assumed all to be of the same dimensions, and the equation is homogeneous, the substitution:

$$y = e^{\int u dx},$$

will result in an equation in u and x of an order less by unity than the given equation.

8.310 Exact differential equations. A linear differential equation:

$$P_n \frac{d^ny}{dx^n} + P_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_1\frac{dy}{dx} + P_0 = P,$$

where $P, P_0, P_1, \ldots P_n$ are functions of x is exact if:

$$P_0 - \frac{dP_1}{dx} + \frac{d^2P_2}{dx^2} - \ldots + (-1)^n\frac{d^nP_n}{dx^n} = 0.$$

The first integral is:

$$Q_n \frac{d^{n-1}y}{dx^{n-1}} + Q_{n-1}\frac{d^{n-2}y}{dx^{n-2}} + \dots + Q_1 y = \int V\,dx + c_1,$$

where,

$$Q_n = P_n,$$
$$Q_{n-1} = P_{n-1} - \frac{dP_n}{dx},$$
$$Q_{n-2} = P_{n-2} - \frac{dP_{n-1}}{dx} + \frac{d^2P_n}{dx^2},$$
$$\dots\dots$$
$$\dots\dots$$
$$Q_1 = P_1 - \frac{dP_2}{dx} + \frac{d^2P_3}{dx^2} - \dots + (-1)^{n-1}\frac{d^{n-1}P_n}{dx^{n-1}}.$$

If the first integral is an exact differential equation the process may be continued as long as the coefficients of each successive integral satisfy the condition of integrability.

8.311 Non-linear differential equations. A non-linear differential equation of the nth order:

$$V\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y, x\right) = 0,$$

to be exact must contain $\frac{d^ny}{dx^n}$ in the first degree only. Put

$$\frac{d^{n-1}y}{dx^{n-1}} = p, \quad \frac{d^ny}{dx^n} = \frac{dp}{dx}.$$

Integrate the equation on the assumption that p is the only variable and $\frac{dp}{dx}$ its differential coefficient. Let the result be V_1. In $V\,dx - dV_1$, $\frac{d^{n-1}y}{dx^{n-1}}$ is the highest differential coefficient and it occurs in the first degree only. Repeat this process as often as may be necessary and the first integral of the exact differential equation will be

$$V_1 + V_2 + \dots = c.$$

If this process breaks down owing to the appearance of the highest differential coefficient in a higher degree than the first the given differential equation was not exact.

8.312 General condition for an exact differential equation. Write:

$$\frac{dy}{dx} = y' \cdot \frac{d^2y}{dx^2} = y'' \ldots\ldots \quad \frac{d^ny}{dx^n} = y^{(n)}.$$

In order that the differential equation:

$$V(x, y, y', y'', \ldots\ldots, y^{(n)}) = 0,$$

be exact it is necessary and sufficient that

$$\frac{\partial V}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial V}{\partial y''}\right) - \ldots\ldots + (-1)^n \frac{\partial^n}{\partial x^n}\left(\frac{\partial V}{\partial y^{(n)}}\right) = 0.$$

8.400 Linear differential equations of the second order.

General form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

where P, Q, R are, in general, functions of x.

8.401 If a solution of the equation with $R = 0$:

$$y = w$$

can be found, the complete solution of the given differential equation is:

$$y = c_2 w + c_1 w \int e^{-\int P dx} \frac{dx}{w^2} + w \int e^{-\int P dx} \frac{dx}{w^2} \int w R e^{\int P dx} dx.$$

8.402 The general linear differential equation of the second order may b reduced to the form:

$$\frac{d^2v}{dx^2} + Iv = Re^{\frac{1}{2}\int P dx},$$

where:

$$y = ve^{-\frac{1}{2}\int P dx},$$

$$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2.$$

8.403 The differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

by the change of independent variable to

$$z = \int e^{-\int P dx} dx,$$

becomes:

$$\frac{d^2y}{dz^2} + Qe^{2\int P dx} y = 0.$$

By the change of independent variable.

$$dz = Qe^{\int P dx} dx,$$

$$Qe^{2 \quad P dx} = \frac{1}{U(z)},$$

it becomes:

$$\frac{d}{dz}\left\{\frac{1}{U}\frac{dy}{dz}\right\} + y = 0.$$

8.404 Resolution of the operator. The differential equation:

$$u\frac{d^2y}{dx^2} + v\frac{dy}{dx} + wy = 0,$$

may sometimes be solved by resolving the operator,

$$u\frac{d^2}{dx^2} + v\frac{d}{dx} + w,$$

into the product,

$$\left(p\frac{d}{dx} + q\right)\left(r\frac{d}{dx} + s\right).$$

The solution of the differential equation reduces to the solution of

$$r\frac{dy}{dx} + sy = c_1 e^{-\int\frac{q}{p}dx}.$$

The equations for determining p, r, q, s are:

$$pr = u,$$

$$qr + ps + p\frac{dr}{dx} = v,$$

$$qs + p\frac{ds}{dx} = w.$$

8.410 Variation of parameters. The complete solution of the differential equation:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R,$$

is

$$y = c_1 f_2(x) + c_2 f_1(x) + \frac{1}{C}\int^x R(\xi)e^{\int^\xi P dx}\left\{ f_2(x)f_1(\xi) - f_1(x)f_2(\xi) \right\} d\xi,$$

where $f_1(x)$ and $f_2(x)$ are two particular solutions of the differential equation with $R = 0$, and are therefore connected by the relation

$$f_1\frac{df_2}{dx} - f_2\frac{df_1}{dx} = Ce^{-\int P dx}.$$

C is an absolute constant depending upon the forms of f_1 and f_2 and may be taken as unity.

8.500 The differential equation:

$$(a_2 + b_2x)\frac{d^2y}{dx^2} + (a_1 + b_1x)\frac{dy}{dx} + (a_0 + b_0x)y = 0.$$

8.501 Let

$$D = (a_0b_1 - a_1b_0)(a_1b_2 - a_2b_1) - (a_0b_2 - a_2b_0)^2.$$

Special cases.

8.502 $b_2 = b_1 = b_0 = 0$.

The solution is:

$$y_1 = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x},$$

where:

$$\frac{\lambda_1}{\lambda_2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}}{2a_2}.$$

8.503 $D = 0,\ b_2 = 0,$

$$y = e^{\lambda x}\left\{ c_1 + c_2 \int e^{-(k+2\lambda)x - mx^2} dx \right\},$$

where:

$$k = \frac{a_1}{a_2} \quad m = \frac{b_1}{2a_2} \quad \lambda = -\frac{b_0}{b_1}.$$

8.504 $D = 0,\ b_2 \neq 0$:

$$y = e^{\lambda x}\left\{ c_1 + c_2 \int e^{-(k+2\lambda)x}(a_2 + b_2 x)^m dx \right\},$$

where

$$k = \frac{b_1}{b_2} \quad m = \frac{a_2 b_1 - a_1 b_2}{b_2^3},$$

and λ is the common root of:

$$a_2\lambda^2 + a_1\lambda + a_0 = 0,$$
$$b_2\lambda^2 + b_1\lambda + b_0 = 0.$$

8.505 $D \neq 0,\ b_2 = b_1 = 0$. If $\eta = f(\xi)$ is the complete solution of:

$$\frac{d^2\eta}{d\xi^2} + \xi\eta = 0,$$

$$y = e^{\lambda x} f\left(\frac{\alpha + \beta x}{\beta^{\frac{2}{3}}}\right),$$

where

$$\alpha = \frac{4a_0 a_2 - a_1^2}{4a_2^2} \quad \beta = \frac{b_0}{a_2} \quad \lambda = -\frac{a_1}{2a_2}.$$

8.510 The differential equation **8.500** under the condition $D \neq 0$ can always be reduced to the form:

$$\xi \frac{d^2\phi}{d\xi^2} + (p + q + \xi)\frac{d\phi}{d\xi} + p\phi = 0.$$

8.511 Denote the complete solution of **8.510**:

$$\phi = F\{\xi\}.$$

8.512 $b_2 = b_1 = 0$:

$$y = e^{\lambda x + (\mu + \nu x)^{\frac{3}{2}}} F\{2(\mu + \nu x)^{\frac{3}{2}}\},$$

where:

$$\lambda = -\frac{a_1}{2a_2} \quad \mu = \frac{a_1^2 - 4a_0 a_2}{4a_2^2}\left(\frac{4a_2^2}{9b_0^2}\right)^{\frac{1}{3}},$$

$$\nu = -\left(\frac{4b_0}{9a_2}\right)^{\frac{1}{3}},$$

$$p = q = \frac{1}{6}.$$

8.513 $b_2 = 0,\ b_1 \neq 0$:

$$y = e^{\lambda x} F\left\{\frac{(\alpha_1 + \beta_1 x)^2}{2\beta_1}\right\},$$

where:

$$\lambda = -\frac{b_0}{b_1}\quad \alpha_1 = \frac{a_1 b_1 - 2a_2 b_0}{a_2 b_1},\quad \beta_1 = \frac{b_1}{a_2},$$

$$p = \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2}{2b_1^3},$$

$$q = \frac{1}{2} - p.$$

8.514 $b_2 \neq 0,\ b_0 = \dfrac{b_1^2}{4b_2}$:

$$y = e^{\lambda x + \sqrt{\mu + \nu x}} F\left\{2\sqrt{\mu + \nu x}\right\},$$

where:

$$\lambda = -\frac{b_1}{2b_2},\quad \mu = -a_2\frac{4a_0 b_2^2 - 2a_1 b_1 b_2 + a_2 b_1^2}{b_2^4},$$

$$\nu = -\frac{4a_0 b_2^2 - 2a_1 b_1 b_2 + a_2 b_1^2}{b_2^3},$$

$$p = q = \frac{a_1 b_2 - a_2 b_1}{b_2^2} - \frac{1}{2}.$$

8.515 $b_2 \neq 0,\ b_0 \neq \dfrac{b_1^2}{4b_2}$:

$$y = e^{\lambda x} F\left\{\frac{\beta_1(\alpha_2 + \beta_2 x)}{\beta_2^2}\right\},$$

where $\alpha_2 = a_2,\ \beta_2 = b_2,\ \beta_1 = 2b_2\lambda + b_1$ and λ is one of the roots of

$$b_2\lambda^2 + b_1\lambda + b_0 = 0.$$

$$p = \frac{a_2\lambda^2 + a_1\lambda + a_0}{2b_2\lambda + b_1},\qquad q = \frac{a_1 b_2 - a_2 b_1}{b_2^2} - p.$$

8.520 The solution of **8.510** will be denoted:

$$\phi = F(p, q, \xi).$$

1. $F(p, q, \xi) = e^{-\xi} F(q, p, -\xi).$
2. $F(p, q, -\xi) = e^{\xi} F(q, p, \xi)$
3. $F(q, p, \xi) = e^{-\xi} F(p, q, -\xi).$
4. $F(p, q, \xi) = \xi^{1-p-q} F(1-q, 1-p, \xi).$
5. $F(-p, -q, \xi) = \xi^{1+p+q} F(1+q, 1+p, \xi).$
6. $F(p+m, q, \xi) = \dfrac{d^m}{d\xi^m} F(p, q, \xi).$
7. $F(p, q+n, \xi) = (-1)^n e^{-\xi}\dfrac{d^n}{d\xi^n}\left\{e^{\xi} F(p, q, \xi)\right\}.$

8.521 The function $F(p, q, \xi)$ can always be found if it is known for positive proper fractional values of p and q.

8.522 p and q positive improper fractions:

$$p = m + r, \quad q = n + s$$

where m and n are positive integers and r and s positive proper fractions.

$$F(m+r, n+s, \xi) = (-1)^n \frac{d^m}{d\xi^m}\left[e^{-\xi} \frac{d^n}{d\xi^n}\left\{ e^{\xi} F(r, s, \xi) \right\}\right].$$

8.523 p and q both negative:

$$p = -(m - 1 + r) \quad q = -(n - 1 + s),$$

$$F(-m+1-r, -n+1-s, \xi) = (-1)^m \xi^{m+n+r+s-1} \frac{d^n}{d\xi^n}\left[e^{-\xi} \frac{d^m}{d\xi^m} \left\{ e^{\xi} F(s, r, \xi) \right\}\right].$$

8.524 p positive, q negative:

$$p = m + r, \quad q = -n + s,$$

$$F(m+r, -n+s, \xi) = \frac{d^m}{d\xi^m}\left[\xi^{n+1-r-s} \frac{d^n}{d\xi^n} F(1-s, 1-r, \xi)\right].$$

8.525 p negative, q positive:

$$p = -m + r, \quad q = n + s,$$

$$F(-m+r, n+s, \xi) = (-1)^{m+n} e^{-\xi} \frac{d^n}{d\xi^n}\left[\xi^{m+1-r-s} \frac{d^m}{d\xi^m}\left\{ e^{\xi} F(1-s, 1-r, \xi) \right\}\right].$$

8.530 If either p or q is zero the relation $D = 0$ is satisfied and the complete solution of the differential equation is given in **8.502, 3.**

8.531 If $p = m$, a positive integer:

$$\phi = F(m, q, \xi) = c_1 \frac{d^{m-1}}{d\xi^{m-1}}\left[\xi^{-q} e^{-\xi} \int \xi^{q-1} e^{\xi} d\xi\right] + c_2 \frac{d^{m-1}}{d\xi^{m-1}}\left[\xi^{-q} e^{-\xi}\right].$$

8.532 If $p = m$, a positive integer and both q and ξ are positive:

$$\phi = F(m, q, \xi) = c_1 \int_0^1 u^{m-1}(1-u)^{q-1} e^{-\xi u}\, du + c_2 e^{-\xi} \int^{\infty} (1+u)^{m-1} u^{q-1} e^{-\xi u}\, du.$$

8.533 If $q = n$, a positive integer:

$$\phi = F(p, n, \xi) = c_1 e^{-\xi} \frac{d^{n-1}}{d\xi^{n-1}}\left[\xi^{-p} e^{\xi} \int \xi^{p-1} e^{-\xi} d\xi\right] + c_2 e^{-\xi} \frac{d^{n-1}}{d\xi^{n-1}}\left[\xi^{-p} e^{\xi}\right].$$

8.534 If $q = n$, a positive integer and both p and ξ are positive:

$$\phi = F(p, n, \xi) = c_1 \int_0^1 u^{p-1}(1-u)^{n-1} e^{-\xi u}\, du + c_2 e^{-\xi} \int_0^{\infty} (1+u)^{p-1} u^{n-1} e^{-\xi u}\, du.$$

8.540 The general solution of equation **8.510** may be written:

$$\phi = F(p, q, \xi) = c_1 M + c_2 N,$$

$$M = \int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du \qquad \begin{matrix} p > 0 \\ q > 0 \end{matrix}$$

$$N = \int_0^\infty (1+u)^{p-1}u^{q-1}e^{-\xi(1+u)}du \qquad \begin{matrix} q > 0 \\ \xi > 0 \end{matrix}$$

$$M = \frac{\Gamma(p)\Gamma(q)}{\Gamma(s)}\left\{1 - \frac{p}{s}\frac{\xi}{1!} + \frac{p(p+1)}{s(s+1)}\frac{\xi^2}{2!} - \frac{p(p+1)(p+2)}{s(s+1)(s+2)}\frac{\xi^3}{3!} + \ldots\ldots\right\}$$

$$s = p + q,$$

$$N = \frac{\Gamma(q)e^{-\xi}}{\xi^q}\left\{1 + \frac{(p-1)q}{1!\xi} + \frac{(p-1)(p-2)q(q+1)}{2!\xi^2} + \ldots\ldots\right.$$

$$+ \frac{(p-1)(p-2)\ldots\ldots(p-n-1)(q)(q+1)\ldots\ldots(q+n-2)}{(n-1)!\xi^{n-1}}$$

$$\left. + \frac{\rho(p-1)(p-2)\ldots\ldots(p-n)q(q+1)(q+2)\ldots\ldots(q+n-1)}{n!\xi^n}\right\},$$

where $0 < \rho < 1$ and the real part of ξ is positive.

THE COMPLETE SOLUTION OF EQUATION **8.510** IN SPECIAL CASES

8.550 $p>0$, $q>0$, real part of $\xi>0$:

$$F(p, q, \xi) = c_1\int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du + c_2e^{-\xi}\int_0^\infty (1+u)^{p-1}u^{q-1}e^{-\xi u}du.$$

8.551 $p>0$, $q>0$, $\xi<0$:

$$F(p, q, \xi) = c_1\int_0^1 u^{p-1}(1-u)^{q-1}e^{-\xi u}du + c_2\int_0^\infty u^{p-1}(1+u)^{q-1}e^{\xi u}du.$$

8.552 $p<0$, $q<0$, $\xi>0$:

$$F(p,q,\xi) = \xi^{1-p-q}\left\{c_1\int_0^1 (1-u)^{-p}u^{-q}e^{-\xi u}du + c_2e^{-\xi}\int_0^\infty u^{-p}(1+u)^{-q}e^{-\xi u}du\right\}.$$

.553 $p<0$, $q<0$, $\xi<0$:

$$, q, \xi) = \xi^{1-p-q}\left\{c_1\int_0^1 (1-u)^{-p}u^{-q}e^{-\xi u}du + c_2\int_0^\infty (1+u)^{-p}u^{-q}e^{+\xi u}du\right\}.$$

$p>0$, $q<0$

$p = m + r$, where m is a positive integer and r a proper fraction.

$$F(m+r, q, \xi) = \frac{d^m}{d\xi^m}\left\{\xi^{1-r-q}F(1-r, 1-q, \xi)\right\},$$

$$\xi > 0:\quad F(1-r,\, 1-q,\, \xi) = c_1 \int_0^1 u^{-r}(1-u)^{-q} e^{-\xi u}\, du + c_2 e^{-\xi} \int_0^\infty (1+u)^{-r} u^{-q} e^{-\xi u}\, du,$$

$$\xi < 0:\quad F(1-r,\, 1-q,\, \xi) = c_1 \int_0^1 u^{-r}(1-u)^{-q} e^{-\xi u}\, du + c_2 \int_0^\infty u^{-r}(1+u)^{-q} e^{\xi u}\, du.$$

8.555 $p < 0,\ q > 0,$

$q = n + s$, where n is a positive integer and s a proper fraction.

$$F(p,\, n+s,\, \xi) = e^{-\xi} \frac{d^n}{d\xi^n} \left\{ e^{\xi} \xi^{1-p-s} F(1-s,\, 1-p,\, \xi) \right\},$$

$$\xi > 0:\quad F(1-s,\, 1-p,\, \xi) = c_1 \int_0^1 u^{-s}(1-u)^{-p} e^{-\xi u}\, du + c_2 e^{-\xi} \int_0^\infty (1+u)^{-s} u^{-p} e^{-\xi u}\, du,$$

$$\xi < 0:\quad F(1-s,\, 1-p,\, \xi) = c_1 \int_0^1 u^{-s}(1-u)^{-p} e^{-\xi}\, du + c_2 \int_0^\infty u^{-s}(1+u)^{-p} e^{\xi u}\, du.$$

8.556 ξ pure imaginary:

$p = r,\ q = s$, where r and s are positive proper fractions.

$r + s \neq 1$:

$$F(r,\, s,\, \xi) = c_1 \int_0^1 u^{r-1}(1-u)^{s-1} e^{-\xi u}\, du + c_2 \xi^{1-r-s} \int_0^1 u^{-s}(1-u)^{-r} e^{-\xi u}\, du.$$

$r + s = 1$:

$$F(r,\, s,\, \xi) = c_1 \int_0^1 u^{r-1}(1-u)^{s-1} e^{-\xi u}\, du + c_2 \int_0^1 u^{r-1}(1-u)^{s-1} e^{-\xi u} \log\left\{ \xi u(1-u) \right\} du.$$

8.600 The differential equation:

$$x \frac{d^2y}{dx^2} + (\gamma - x)\frac{dy}{dx} - \alpha y = 0$$

is satisfied by the confluent hypergeometric function. The complete solution is:

$$y = c_1 M(\alpha,\, \gamma,\, x) + c_2 x^{1-\gamma} M(\alpha - \gamma + 1,\, 2 - \gamma,\, x) = \overline{M}(\alpha,\, \gamma,\, x),$$

where

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!} + \cdots\cdots$$

The series is absolutely and uniformly convergent for all real and complex values of α, γ, x, except when γ is a negative integer or zero.

When γ is a positive integer the complete solution of the differential equation is:

$$y = \left\{c_1 + c_2 \log x\right\} M(\alpha, \gamma, x) + c_2 \left\{ \frac{\alpha x}{\gamma}\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right) \right.$$

$$+ \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - 1 - \frac{1}{2}\right)$$

$$+ \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}\left(\frac{1}{\alpha} + \frac{1}{\alpha+1} + \frac{1}{\alpha+2} - \frac{1}{\gamma} - \frac{1}{\gamma+1} - \frac{1}{\gamma+2} - 1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$\left. + \cdots\cdots \right\}.$$

8.601 For large values of x the following asymptotic expansion may be used:

$$M(\alpha, \gamma, x)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-x)^{-\alpha}\left\{1 - \frac{\alpha(\alpha-\gamma+1)}{1}\frac{1}{x} + \frac{\alpha(\alpha+1)(\alpha-\gamma+1)(\alpha-\gamma+2)}{2!}\frac{1}{x^2}\cdots\right\}$$

$$+ \frac{\Gamma(\gamma)}{\Gamma(\alpha)}e^x x^{\alpha-\gamma}\left\{1 + \frac{(1-\alpha)(\gamma-\alpha)}{1}\frac{1}{x} + \frac{(1-\alpha)(2-\alpha)(\gamma-\alpha)(\gamma-\alpha+1)}{2!}\frac{1}{x^2} + \cdots\right\}.$$

8.61

1. $M(\alpha, \gamma, x) = e^x M(\gamma - \alpha, \gamma, -x)$.
2. $x^{1-\gamma} M(\alpha - \gamma + 1, 2 - \gamma, x) = e^x x^{1-\gamma} M(1 - \alpha, 2 - \gamma, -x)$.
3. $\dfrac{x}{\gamma} M(\alpha + 1, \gamma + 1, x) = M(\alpha + 1, \gamma, x) - M(\alpha, \gamma, x)$.
4. $\alpha M(\alpha + 1, \gamma + 1, x) = (\alpha - \gamma) M(\alpha, \gamma + 1, x) + \gamma M(\alpha, \gamma, x)$.
5. $(\alpha + x) M(\alpha + 1, \gamma + 1, x) = (\alpha - \gamma) M(\alpha, \gamma + 1, x) + \gamma M(\alpha + 1, \gamma, x)$.
6. $\alpha\gamma M(\alpha + 1, \gamma, x) = \gamma(\alpha + x) M(\alpha, \gamma, x) - x(\gamma - \alpha) M(\alpha, \gamma + 1, x)$.
7. $\alpha M(\alpha + 1, \gamma, x) = (x + 2\alpha - \gamma) M(\alpha, \gamma, x) + (\gamma - \alpha) M(\alpha - 1, \gamma, x)$.
8. $\dfrac{\gamma - \alpha}{\gamma} x M(\alpha, \gamma + 1, x) = (x + \gamma - 1) M(\alpha, \gamma, x) + (1 - \gamma) M(\alpha, \gamma - 1, x)$.

8.62

1. $\dfrac{d}{dx} M(\alpha, \gamma, x) = \dfrac{\alpha}{\gamma} M(\alpha + 1, \gamma + 1, x)$.
2. $(1 - \alpha)\displaystyle\int_0^x M(\alpha, \gamma, x)\, dx = (1 - \gamma) M(\alpha - 1, \gamma - 1, x) + (\gamma - 1)$.

SPECIAL DIFFERENTIAL EQUATIONS AND THEIR SOLUTIONS IN TERMS OF $\overline{M}(\alpha, \gamma, x)$

8.630

$$\frac{d^2y}{dx^2} + 2(p + qx)\frac{dy}{dx} + \left\{ 4\alpha q + p^2 - q^2m^2 + 2qx(p + qm) \right\} y = 0,$$

$$y = e^{-(p+qm)x}\,\overline{M}\left(\alpha,\ \frac{1}{2},\ -q(x - m)^2\right).$$

8.631

$$\frac{d^2y}{dx^2} + \left(2p + \frac{\gamma}{x}\right)\frac{dy}{dx} + \left\{ p^2 - l^2 + \frac{1}{x}\left(\gamma p + \gamma l - 2\alpha l\right) \right\} y = 0,$$

$$y = e^{-(p+l)x}\,\overline{M}(\alpha,\ \gamma,\ 2lx).$$

8.632

$$\frac{d^2y}{dx^2} + 2(p + qx)\frac{dy}{dx} + \left\{ q + c(1 - 4\alpha) + (p + qx)^2 - c^2(x - m)^2 \right\} y = 0,$$

$$y = e^{-px-\frac{1}{2}qx^2-\frac{1}{2}c(x-m)^2}\,\overline{M}\left(\alpha,\ \frac{1}{2},\ c(x - m)^2\right).$$

8.633

$$\frac{d^2y}{dx^2} + \left(2p + \frac{q}{x}\right)\frac{dy}{dx} + \left\{ p^2 - l^2 + \frac{1}{x}\left(pq + \gamma l - 2\alpha l\right) + \frac{1}{4x^2}\left(\gamma - q\right)\left(2 - q - \gamma\right) \right\} y = 0,$$

$$y = e^{-(p+l)x}\,x^{\frac{\gamma-q}{2}}\,\overline{M}\,(\alpha,\ \gamma,\ 2lx).$$

8.634

$$\frac{d^2y}{dx^2} + \left\{ \frac{2\gamma - 1}{x} + 2\alpha + 2(b - c)x \right\} \frac{dy}{dx}$$

$$+ \left\{ \frac{\alpha(2\gamma - 1)}{x} + (a^2 + 2b\gamma - 4\alpha c) + 2a(b - c)x + b(b - 2c)x^2 \right\} y = 0,$$

$$y = e^{-ax-\frac{1}{2}bx^2}\,\overline{M}(\alpha,\ \gamma,\ cx^2).$$

8.635

$$\frac{d^2y}{dx^2} + \frac{1}{x}\left(2px^r + qr - r + 1\right)\frac{dy}{dx}$$

$$+ \frac{1}{x^2}\left\{ (p^2 - l^2)x^{2r} + r(pq + \gamma l - 2\alpha l)x^r + \frac{1}{4}r^2(\gamma - q)(2 - q - \gamma) \right\} y = 0,$$

$$y = e^{-\frac{(p+l)}{r}x^r}\,x^{\frac{r}{2}(\gamma-q)}\,\overline{M}\left(\alpha,\ \gamma,\ \frac{2lx^r}{r}\right).$$

tions of any of these differential equations. The range in x is 1 to 10; in α, $+0.5$ to $+4.0$ and -0.5 to -3.0; in γ, 1 to 7. For negative values of x the equations of **8.61** may be used.

SPECIAL DIFFERENTIAL EQUATIONS

8.700

$$\frac{d^2y}{dx^2} + n^2 y = X(x)$$

where $X(x)$ is any function of x. The complete solution is:

$$y = c_1 e^{nx} + c_2 e^{-nx} + \frac{1}{n}\int^x X(\xi)\,\sinh n(x-\xi)\,d\xi.$$

8.701

$$\frac{d^2y}{dx^2} + \kappa\frac{dy}{dx} + n^2 y = X(x).$$

The complete solution, satisfying the conditions:

$$x = 0 \qquad y = y_0,$$

$$x = 0 \qquad \frac{dy}{dx} = y_0{}',$$

$$y = e^{-\frac{1}{2}\kappa x}\left\{ y_0{}'\frac{\sin n'x}{n'} + y_0\left(\cos n'x + \frac{\kappa}{2n'}\,\sin n'x\right)\right\}$$

$$+\frac{1}{n'}\int_0^x e^{-\frac{1}{2}\kappa(x-\xi)}\sin n'(x-\xi)X(\xi)\,d\xi,$$

where
$$n' = \sqrt{n^2 - \frac{\kappa^2}{4}}.$$

8.702

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(x)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$y = \int\frac{e^{-\int f(x)dx}\,dx}{\int e^{-\int f(x)dx}g(x)\,dx + c_1} + c_2.$$

8.703

$$\frac{d^2y}{dx^2} + f(y)\left(\frac{dy}{dx}\right)^2 + g(y) = 0,$$

$$x = \pm\int\frac{e^{\int f(y)dy}\,dy}{\{c_1 - 2\int e^{2\int f(y)dy}g(y)\,dy\}^{\frac{1}{2}}} + c_2.$$

8.704

$$\frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$x = \int\frac{e^{\int g(y)dy}\,dy}{c_1 - \int e^{\int g(y)dy}f(y)\,dy} + c_2.$$

8.705

$$\frac{d^2y}{dx^2} + f(x)\frac{dy}{dx} + g(y)\left(\frac{dy}{dx}\right)^2 = 0,$$

$$\int e^{\int g(y)dy}\,dy = c_1\int e^{-\int f(x)dx}\,dx + c_2.$$

8.706

$$\frac{d^2y}{dx^2} + (a + bx)\frac{dy}{dx} + abxy = 0.$$

$$y = e^{-ax}\left\{c_1 + c_2\int e^{ax - \frac{1}{2}bx^2}\,dx\right\}.$$

8.707

$$x\frac{d^2y}{dx^2} + (a + bx)\frac{dy}{dx} + aby = 0,$$

$$y = e^{-bx}\left\{c_1 + c\int x^{-a}e^{bx}\,dx\right\}.$$

8.708

$$\frac{d^2y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{b}{x^2}\,y = 0.$$

1. $(a - 1)^2 > 4b$; $\lambda = \frac{1}{2}\sqrt{(a-1)^2 - 4b}$

$$y = x^{-\frac{a-1}{2}}\left\{c_1x^{\lambda} + c_2x^{-\lambda}\right\}.$$

2. $(a - 1)^2 < 4b$; $\lambda = \frac{1}{2}\sqrt{4b - (a-1)^2}$

$$y = x^{-\frac{a-1}{2}}\left\{c_1\cos(\lambda\log x) + c_2\sin(\lambda\log x)\right\}.$$

3. $(a - 1)^2 = 4b$

$$y = x^{-\frac{a-1}{2}}(c_1 + c_2\log x).$$

8.709

$$\frac{d^2y}{dx^2} + 2bx\frac{dy}{dx} + (a + b^2x^2\, y = 0.$$

1. $a < b$, $\lambda = \sqrt{b - a}$,

$$y = e^{-\frac{bx^2}{2}}(c_1e^{\lambda x} + c_2e^{-\lambda x}).$$

2. $a > b$, $\lambda = \sqrt{a - b}$,

$$y = e^{-\frac{bx^2}{2}}(c_1\cos\lambda x + c_2\sin\lambda x).$$

8.710

$$f(x)\frac{d^2y}{dx^2} - (a + bx)\frac{dy}{dx} + by = 0,$$

$$\int\frac{a + bx}{f(x)}\,dx = X,$$

$$y = c_1(a + bx) + c_2\left\{e^X - (a + bx)\int\frac{1}{f(x)}e^X\,dx\right\}.$$

8.711

$$(a^2 - x^2)\frac{d^2y}{dx^2} + 2(\mu - 1)x\frac{dy}{dx} - \mu(\mu - 1)y = 0,$$

$$y = (a + x)^\mu \left\{ c_1 + c_2 \int \frac{(a - x)^{\mu - 1}}{(a + x)^{\mu + 1}} dx \right\}.$$

8.712

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \mu^2 y = \frac{a}{x},$$

$$y = \frac{1}{x}\left\{ c_1 \cos \mu x + c_2 \sin \mu x + \frac{a}{\mu^2} \right\}.$$

8.713

$$\frac{d^4y}{dx^4} + 2d\frac{d^3y}{dx^3} + c\frac{d^2y}{dx^2} + 2b\frac{dy}{dx} + ay = 0,$$

$$y = c_1 e^{-p_1 x}\{p_1 \sin(\omega_1 x + \alpha_1) + \omega_1 \cos(\omega_1 x + \alpha_1)\}$$
$$+ c_2 e^{-p_2 x}\{p_2 \sin(\omega_2 x + \alpha_2) + \omega_2 \cos(\omega_2 x + \alpha_2)\},$$

where:

$$4\omega_1^2 = z + c - 2d^2 + 2\sqrt{z^2 - 4a} - 2d\sqrt{z - c + d^2},$$
$$4\omega_2^2 = z + c - 2d^2 - 2\sqrt{z^2 - 4a} + 2d\sqrt{z - c + d^2},$$
$$2p_1 = d + \sqrt{z - c + d^2},$$
$$2p_2 = d - \sqrt{z - c + d^2},$$

and z is a root of

$$z^3 - cz^2 - 4(a - bd)z + 4(ac - ad^2 - b^2) = 0.$$

(Kiebitz, Ann. d. Physik, 40, p. 138, 1913)

IX. DIFFERENTIAL EQUATIONS
(*continued*)

9.00 Legendre's Equation:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

9.001 If n is a positive integer one solution is the Legendre polynomial, or Zonal Harmonic, $P_n(x)$:

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \dots\right\}.$$

9.002 If n is even the last term in the finite series in the brackets is:

$$(-1)^{\frac{n}{2}}\frac{(n!)^2}{\left(\frac{n}{2}!\right)^2(2n)!}.$$

9.003 If n is odd the last term in the brackets is:

$$(-1)^{\frac{n-1}{2}}\frac{(n!)^2(n-1)!}{([\frac{1}{2}(n-1)]!)^2\,(2n-1)!}x.$$

9.010 If n is a positive integer a second solution of Legendre's Equation is the infinite series:

$$Q_n(x) = \frac{2^n(n!)^2}{(2n+1)!}\left\{x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}x^{-(n+5)} + \dots\right\}.$$

9.011

$$P_{2n}(\cos\theta) = (-1)^n\frac{(2n)!}{2^{2n}(n!)^2}\left\{\sin^{2n}\theta - \frac{(2n)^2}{2!}\sin^{2n-2}\theta\cos^2\theta + \dots + (-1)^n\frac{(2n)^2(2n-2)^2\dots 4^2 2^2}{(2n)!}\cos^{2n}\theta\right\}.$$

9.012

$$P_{2n+1}(\cos\theta) = (-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}\left\{\sin^{2n}\theta\cos\theta - \frac{(2n)^2}{3!}\sin^{2n-2}\theta\cos^3\theta + \dots + (-1)^n\frac{(2n)^2(2n-2)^2\dots 4^2 2^2}{(2n+1)!}\cos^{2n+1}\theta\right\}.$$

(Brodetsky: Mess. of Math. 42, p. 65, 1912)

9.02 Recurrence formulae for $P_n(x)$:

1. $$(n+1)P_{n+1} + nP_{n-1} = (2n+1)xP_n.$$

2. $$(2n+1)P_n = \frac{dP_{n+1}}{dx} - \frac{dP_{n-1}}{dx}.$$

3. $$(n+1)P_n = \frac{dP_{n+1}}{dx} - x\frac{dP_n}{dx}.$$

4. $$nP_n = x\frac{dP_n}{dx} - \frac{dP_{n-1}}{dx}.$$

5. $$(1-x^2)\frac{dP_n}{dx} = (n+1)(xP_n - P_{n+1}).$$

6. $$(1-x^2)\frac{dP_n}{dx} = n(P_{n-1} - xP_n).$$

7. $$(2n+1)(1-x^2)\frac{dP_n}{dx} = n(n+1)(P_{n-1} - P_{n+1}).$$

9.028 Recurrence formulae for $Q_n(x)$. These are the same as those for $P_n(x)$.

9.030 Special Values.

$$P_0(x) = 1,$$
$$P_1(x) = x,$$
$$P_2(x) = \tfrac{1}{2}(3x^2 - 1),$$
$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3),$$
$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x),$$
$$P_6(x) = \tfrac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5),$$
$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x),$$
$$P_8(x) = \tfrac{1}{128}(6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35).$$

9.031

$$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1},$$

$$Q_1(x) = \frac{1}{2}x\log\frac{x+1}{x-1} - 1,$$

$$Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1} - \frac{3}{2}x,$$

9.032

$$P_{2n}(0) = (-1)^n \frac{1\cdot 3\cdot 5 \;.\;.\;.\;.\; (2n-1)}{2\cdot 4\cdot 6 \;.\;.\;.\;.\; 2n},$$

$$P_{2n+1}(0) = 0,$$

$$P_n(1) = 1,$$

$$P_n(-x) = (-1)^n P_n(x).$$

9.033 If $z = r\cos\theta$:

$$\frac{\partial P_n(\cos\theta)}{\partial z} = \frac{n+1}{r}\left\{P_1(\cos\theta)P_n(\cos\theta) - P_{n+1}(\cos\theta)\right\}$$

$$= \frac{n(n+1)}{(2n+1)r}\left\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\right\}.$$

9.034 Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}\,(x^2-1)^n.$$

9.035 If $z = r\cos\theta$:

$$P_n(\cos\theta) = \frac{(-1)^n}{n!}\,r^{n+1}\frac{\partial^n}{\partial z^n}\left(\frac{1}{r}\right).$$

9.036 If $m \leqq n$:

$$P_m(x)P_n(x) = \sum_{k=0}^{m} \frac{A_{m-k}A_kA_{n-k}}{A_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right)P_{n+m-2k}(x),$$

where:

$$A_r = \frac{1\cdot 3\cdot 5 \;.\;.\;.\;.\; (2r-1)}{r!}.$$

MEHLER'S INTEGRALS

9.040 For all values of n:

$$P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos\,(n+\frac{1}{2})\phi d\phi}{\sqrt{2(\cos\phi-\cos\theta)}}.$$

9.041 If n is a positive integer:

$$P_n(\cos\theta) = \frac{2}{\pi}\int_\theta^\pi \frac{\sin\,(n+\frac{1}{2})\phi d\phi}{\sqrt{2(\cos\theta-\cos\phi)}}.$$

LAPLACE'S INTEGRALS, FOR ALL VALUES OF n

9.042

$$P_n(x) = \frac{1}{\pi}\int_0^\pi \{x+\sqrt{x^2-1}\cos\phi\}^n\,d\phi.$$

9.043

$$Q_n(x) = \int_0^\infty \frac{d\phi}{\{x+\sqrt{x^2-1}\cosh\phi\}^{n+1}}.$$

INTEGRAL PROPERTIES

9.044

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = 0 \text{ if } m \neq n$$
$$= \frac{2}{2n+1} \text{ if } m = n.$$

9.045

$$(m-n)(m+n+1)\int_x^1 P_m(x)P_n(x)\, dx$$
$$= \tfrac{1}{2}\{P_m[(n+1)P_{n+1} - nP_{n-1}] - P_n[(m+1)P_{m+1} - mP_{m-1}]\}.$$

9.046

$$(2n+1)\int_x^1 P_n^2(x)\, dx = 1 - xP_n^2 - 2x(P_1^2 + P_2^2 + \dots + P_{n-1}^2)$$
$$+ 2(P_1P_2 + P_2P_3 + \dots + P_{n-1}P_n)$$

EXPANSIONS IN LEGENDRE FUNCTIONS

9.050 Neumann's expansion:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x),$$

$$a_n = (n + \tfrac{1}{2})\int_{-1}^{+1} f(x) P_n(x)\, dx,$$

$$= \frac{n + \frac{1}{2}}{2^n n!}\int_{-1}^{+1} f^{(n)}(x)\cdot(1-x^2)^n\, dx.$$

9.051 Any polynomial in x may be expressed as a series of Legendre's polynomials. If $f_n(x)$ is a polynomial of degree n:

$$f_n(x) = \sum_{k=0}^{n} a_k P_k(x),$$

$$a_k = \frac{2k+1}{2}\int_{-1}^{+1} f_n(x) P_k(x)\, dx.$$

SPECIAL EXPANSIONS IN LEGENDRE FUNCTIONS

9.060 For all positive real values of n:

$$1.\ \cos n\theta = -\frac{1+\cos n\pi}{2(n^2-1)}\left\{P_0(\cos\theta) + \frac{5n^2}{(n^2-3^2)}P_2(\cos\theta)\right.$$
$$\left. + \frac{9n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}P_4(\cos\theta) + \dots\right\} - \frac{1-\cos n\pi}{2(n^2-2^2)}\left\{3P_1(\cos\theta)\right.$$
$$\left. + \frac{7(n^2-1^2)}{(n^2-4^2)}P_3(\cos\theta) + \frac{11(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}P_5(\cos\theta) + \dots\right\}.$$

2. $$\sin n\theta = -\frac{1}{2}\frac{\sin n\pi}{(n^2-1)}\left\{P_0(\cos\theta) + \frac{5n^2}{(n^2-3^2)}P_2(\cos\theta) + \frac{9n^2(n^2-2^2)}{(n^2-3^2)(n^2-5^2)}P_4(\cos\theta)+\ldots\right\} + \frac{1}{2}\frac{\sin n\pi}{(n^2-2^2)}\left\{3P_1(\cos\theta) + \frac{7(n^2-1^2)}{(n^2-4^2)}P_3(\cos\theta) + \frac{11(n^2-1^2)(n^2-3^2)}{(n^2-4^2)(n^2-6^2)}P_5(\cos\theta)+\ldots\right\}.$$

9.061 If n is a positive integer:

1. $$\cos n\theta = \frac{1}{2}\frac{2\cdot4\cdot6\ldots2n}{3\cdot5\cdot7\ldots(2n+1)}\left\{(2n+1)P_n(\cos\theta) + (2n-3)\frac{[n^2-(n+1)^2]}{[n^2-(n-2)^2]}P_{n-2}(\cos\theta) + (2n-7)\frac{[n^2-(n+1)^2][n^2-(n-1)^2]}{[n^2-(n-2)^2][n^2-(n-4)^2]}P_{n-4}(\cos\theta)+\ldots\right\}.$$

2. $$\sin n\theta = \frac{\pi}{4}\frac{1\cdot3\cdot5\ldots(2n-3)}{2\cdot4\cdot6\ldots(2n-2)}\left\{(2n-1)P_{n-1}(\cos\theta) + (2n+3)\frac{[n^2-(n-1)^2]}{[n^2-(n+2)^2]}P_{n+1}(\cos\theta) + (2n+7)\frac{[n^2-(n-1)^2][n^2-(n+1)^2]}{[n^2-(n+2)^2][n^2-(n+4)^2]}P_{n+3}(\cos\theta)+\ldots\right\}.$$

9.062

1. $$\theta = \frac{\pi}{2} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(4n-1)}{(2n-1)^2}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n-1}(\cos\theta).$$

2. $$\sin\theta = \frac{\pi}{4} - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{(4n+1)}{(2n-1)(2n+2)}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n}(\cos\theta).$$

3. $$\cot\theta = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{2n(4n-1)}{(2n-1)}\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n-1}(\cos\theta).$$

4. $$\csc\theta = \frac{\pi}{2} + \frac{\pi}{2}\sum_{n=1}^{\infty}(4n+1)\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^2 P_{2n}(\cos\theta).$$

9.063

1. $$\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = 1 + \sum_{n=1}^{\infty}\frac{1}{n+1}P_n(\cos\theta).$$

2. $$\log\frac{\tan\frac{1}{4}(\pi-\theta)}{\frac{1}{2}\sin\theta} = -\log\sin\frac{\theta}{2} - \log\left(1+\sin\frac{\theta}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}P_n(\cos\theta).$$

9.064 $K(k)$ and $E(k)$ denote the complete elliptic integrals of the fir second kinds, and $k = \sin\theta$:

1. $$K(k) = \frac{\pi^2}{4} + \frac{\pi^2}{4}\sum_{n=1}^{\infty}(-1)^n(4n+1)\left(\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots2n}\right)^3 P_{2n}(\cos\theta).$$

2. $E(k) = \frac{\pi^2}{8} + \frac{\pi^2}{4}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(4n+1)}{(2n-1)(2n+2)}\left(\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\ldots\cdot 2n}\right)^2 P_{2n}(\cos\theta).$

(Hargreaves, Mess. of Math. 26, p. 89, 1897)

9.070 The differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0.$$

If m is a positive integer, and $-1 \geqslant x \geqslant +1$, two solutions of this differential equation are the associated Legendre functions

$$P_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m P_n(x)}{dx^m},$$

$$Q_n^m(x) = (1-x^2)^{\frac{m}{2}}\frac{d^m Q_n(x)}{dx^m}.$$

9.071 If n, m, r are positive integers, and $n \geqslant m$, $r \geqslant m$:

$$\int_{-1}^{+1} P_n^m(x)\, P_r^m(x)\, dx = 0 \text{ if } r \neq n,$$

$$= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \text{ if } r = n.$$

9.100 Bessel's Differential Equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

9.101 One solution is:

$$y = J_\nu(x) = \sum_{k=0}^{\infty}(-1)^k\frac{x^{\nu+2k}}{2^{\nu+2k}k!\,\Gamma(\nu+k+1)}.$$

9.102 A second independent solution when ν is not an integer is:

$$y = J_{-\nu}(x).$$

9.103 If $\nu = n$, an integer:

$$J_{-n}(x) = (-1)^n J_n(x).$$

9.104 A second independent solution when $\nu = n$, an integer, is:

$$Y_n(x) = 2J_n(x)\cdot\log\frac{x}{2} - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n}$$

$$-\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!(k+n)!}\left(\frac{x}{2}\right)^{n+2k}\left\{\psi(k+1)+\psi(k+n+1)\right\}$$

105 For all values of ν, whether integral or not:

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}\Big(\cos \nu\pi J_\nu(x) - J_{-\nu}(x)\Big),$$

$$J_{-\nu}(x) = \cos \nu\pi J_\nu(x) - \sin \nu\pi Y_\nu(x),$$

$$Y_{-\nu}(x) = \sin \nu\pi J_\nu(x) + \cos \nu\pi Y_\nu(x).$$

9.106 For $\nu = n$, an integer:

$$Y_{-n}(x) = (-1)^n Y_n(x).$$

9.107 Cylinder Functions of the third kind, solutions of Bessel's differential equation:

1. $$\overset{\mathrm{I}}{H}_\nu(x) = J_\nu(x) + iY_\nu(x).$$

2. $$\overset{\mathrm{II}}{H}_\nu(x) = J_\nu(x) - iY_\nu(x).$$

3. $$\overset{\mathrm{I}}{H}_{-\nu}(x) = e^{\nu\pi i}\,\overset{\mathrm{I}}{H}_\nu(x).$$

4. $$\overset{\mathrm{II}}{H}_{-\nu}(x) = e^{-\nu\pi i}\,\overset{\mathrm{II}}{H}_\nu(x).$$

9.110 Recurrence formulae satisfied by the functions J_ν, Y_ν, $\overset{\mathrm{I}}{H}_\nu$, $\overset{\mathrm{II}}{H}_\nu$, C_ν represents any one of these functions.

1. $$C_{\nu-1}(x) - C_{\nu+1}(x) = 2\frac{d}{dx}C_\nu(x).$$

2. $$C_{\ -1}(x) + C_{\nu+1}(x) = \frac{2\nu}{x}C_\nu(x).$$

3. $$\frac{d}{dx}C_\nu(x) = C_{\nu-1}(x) - \frac{\nu}{x}C_\nu(x).$$

4. $$\frac{d}{dx}C\ (x) = \frac{\nu}{x}C_\nu(x) - C_{\nu+1}(x).$$

5. $$\frac{d}{dx}\left\{x^\nu C_{\nu(x)}\right\} = x^\nu C_{\nu-1}(x).$$

6. $$\frac{d^2C_\nu(x)}{dx^2} = \frac{1}{4}\left\{C_{\nu+2}(x) + C_{\nu-2}(x) - 2C_\nu(x)\right\}.$$

9.111

1. $J_\nu(x)\dfrac{dY_\nu(x)}{dx} - Y_\nu(x)\dfrac{dJ_\nu(x)}{dx} \quad \pi x \qquad\qquad \pi x$

ASYMPTOTIC EXPANSIONS FOR LARGE VALUES OF x

9.120

1. $$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{P\ (x)\cos\left(x - \frac{2\nu+1}{4}\pi\right) - Q_\nu(x)\sin\left(x - \frac{2\nu+1}{4}\pi\right)\right\},$$

2. $$Y_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{P_\nu(x)\sin\left(x - \frac{2\nu+1}{4}\pi\right) + Q_\nu(x)\cos\left(x - \frac{2\nu+1}{4}\pi\right)\right\},$$

3. $H_\nu^{\mathrm{I}}(x) = e^{i\left(x - \frac{2\nu+1}{4}\pi\right)}\sqrt{\frac{2}{\pi x}}\left\{ P_\nu(x) + iQ_\nu(x) \right\}$,

4. $H_\nu^{\mathrm{II}}(x) = e^{-i\left(x - \frac{2\nu+1}{4}\pi\right)}\sqrt{\frac{2}{\pi x}}\left\{ P_\nu(x) - iQ_\nu(x) \right\}$,

where

$$P_\nu(x) = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\ldots\ldots(4\nu^2 - \overline{4k-1}^2)}{(2k)!\, 2^{6k}\, x^{2k}},$$

$$Q_\nu(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\ldots\ldots(4\nu^2 - \overline{4k-3}^2)}{(2k-1)!\, 2^{6k-3}\, x^{2k-1}}.$$

SPECIAL VALUES

9.130

1. $J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \ldots\ldots$

2. $J_1(x) = -\frac{dJ_0(x)}{dx} = \frac{x}{2}\left\{ 1 - \frac{1}{1!2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{x}{2}\right)^6 + \ldots \right\}$.

3. $$\frac{\pi}{2}Y_0(x) = \left(\log\frac{x}{2} + \gamma\right)J_0(x) + \left(\frac{x}{2}\right)^2 - \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{x}{2}\right)^6 - \ldots$$

$$= \left(\log\frac{x}{2} + \gamma\right)J_0(x) + 4\left\{ \frac{1}{2}J_2(x) - \frac{1}{4}J_4(x) + \frac{1}{6}J_6(x) - \ldots \right\}.$$

4. $$\frac{\pi}{2}Y_1(x) = \left(\log\frac{x}{2} + \gamma\right)J_1(x) - \frac{1}{x}J_0(x) - \frac{x}{2}\left\{ 1 - \frac{1}{1!2!}\left(1 + \frac{1}{2}\right)\left(\frac{x}{2}\right)^2 + \frac{1}{2!3!}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{x}{2}\right)^4 - \ldots\ldots \right\}$$

$$= \left(\log\frac{x}{2} + \gamma\right)J_1(x) - \frac{1}{x}J_0(x) + \frac{3}{1\cdot 2}J_3(x) - \frac{5}{2\cdot 3}J_5(x) + \frac{7}{3\cdot 4}J_7(x) - \ldots$$

$\gamma = 0.5772157$ (6.602).

9.131 Limiting values for $x = 0$:

$$J_0(x) = 1,$$
$$J_1(x) = 0,$$
$$Y_0(x) = \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right),$$
$$Y_1(x) = -\frac{2}{\pi x}.$$

9.132 Limiting values for $x = \infty$:

$$J_0(x) = \frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}, \qquad Y_0(x) = \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}},$$

$$J_1(x) = \frac{\sin\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}, \qquad Y_1(x) = -\frac{\cos\left(x - \frac{\pi}{4}\right)}{\sqrt{\frac{\pi x}{2}}}.$$

9.140 Bessel's Addition Formula:

$$J_\nu(x + h) = \left(\frac{x + h}{x}\right)^\nu \sum_{k=0}^{\infty} (-1)^k \frac{h^k}{k!} \left(\frac{2x + h}{2x}\right)^k J_{\nu+k}(x).$$

9.141 Multiplication formula:

$$J_\nu(\alpha x) = \alpha^\nu \sum_{k=0}^{\infty} \frac{(1 - \alpha^2)^k}{k!} \left(\frac{x}{2}\right)^k J_{\nu+k}(x).$$

9.142

$$J_\nu(\alpha x) J_\mu(\beta x) = \sum_{k=0}^{\infty} (-1)^k A_k \left(\frac{x}{2}\right)^{\mu+\nu+2k},$$

where

$$A_k = \sum_{s=0}^{k} \frac{\alpha^{2k-2s}\beta^{2s}}{s!(k - s)!\Gamma(\nu + k - s + 1)\Gamma(\mu + s + 1)}.$$

9.143

$$J_\nu(x) J_\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1)\Gamma(\mu + k + 1)} \binom{\mu + \nu + 2k}{k} \left(\frac{x}{2}\right)^{\mu+\nu+2k}.$$

DEFINITE INTEGRAL EXPRESSIONS FOR BESSEL'S FUNCTIONS

9.150

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos(x \sin\phi) \cos^{2\nu}\phi \cdot d\phi.$$

9.151

$$J_\nu(x) = \frac{2\left(\frac{x}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\nu + \frac{1}{2}\right)} \int_0^{\pi} \cos(x \cos\phi) \sin^{2\nu}\phi \cdot d\phi.$$

9.152

$$J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^\pi e^{ix\cos\phi}\sin^{2\nu}\phi\cdot d\phi.$$

If n is an integer:

9.153

$$J_{2n}(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\cos(2n\phi)d\phi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}.$$

9.154

$$J_{2n}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \cos(x\cos\phi)\cos(2n\phi)d\phi = \frac{2(-1)^n}{\pi}\int_0^{\frac{\pi}{2}}.$$

9.155

$$J_{2n+1}(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\phi)\sin(2n+1)\phi\,d\phi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}.$$

9.156

$$J_{2n+1}(x) = \frac{(-1)^n}{\pi}\int_0^\pi \sin(x\cos\phi)\cos(2n+1)\phi\,d\phi = \frac{2(-1)^n}{\pi}\int_0^{\frac{\pi}{2}}.$$

9.157

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-in\phi+ix\sin\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi+ix\sin\phi}\,d\phi.$$

INTEGRAL PROPERTIES

9.160 If $C_\nu(\mu x)$ is any one of the particular integrals:

$$J_\nu(\mu x),\ Y_\nu(\mu x),\ H_\nu^{\mathrm{I}}(\mu x),\ H_\nu^{\mathrm{II}}(\mu x),$$

of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(\mu^2 - \frac{\nu^2}{x^2}\right)y = 0,$$

$$\int_a^b C_\nu(\mu_k x)C_\nu(\mu_l x)x\,dx$$

$$= \frac{1}{\mu_k^2-\mu_l^2}\left[x\left\{\mu_l C_\nu(\mu_k x)C_\nu'(\mu_l x) - \mu_k C_\nu(\mu_l x)C_\nu'(\mu_k x)\right\}\right]_a^b;\ \mu_k \neq \mu_l.$$

9.161 If μ_k and μ_l are two different roots of

$$C_\nu(\mu b) = 0,$$

$$\int_a^b C_\nu(\mu_k x)C_\nu(\mu_l x)x\,dx = \frac{a}{\mu_k^2-\mu_l^2}\left\{\mu_k C_\nu(\mu_l a)C_\nu'(\mu_k a) - \mu_l C_\nu(\mu_k a)C_\nu'(\mu_l a)\right\}.$$

9.162 If μ_k and μ_l are two different roots of

$$a\frac{C_\nu'(\mu a)}{C_\nu(\mu a)} = p\mu + q\frac{1}{\mu},$$

and

$$C_\nu(\mu b) = 0,$$

$$\int^b C_\nu(\mu_k x)C_\nu(\mu_l x)x\,dx = pC_\nu(\mu_k a)C_\nu(\mu_l a).$$

If $\mu_k = \mu_l$:

$$\int_a^b C_\nu(\mu_k x)C_\nu(\mu_l x)x\,dx = \frac{1}{2}\left\{b^2C_\nu'^2(\mu_k b) - a^2C_\nu'^2(\mu_k a) - \left(a^2 - \frac{\nu^2}{\mu_k^2}\right)C_\nu^2(\mu_k a)\right\}.$$

EXPANSIONS IN BESSEL'S FUNCTIONS

9.170 Schlömilch's Expansion. Any function $f(x)$ which has a continuous differential coefficient for all values of x in the closed range $(0, \pi)$ may be expanded in the series:

$$f(x) = a_0 + \sum_{k=1} a_k J_0(kx),$$

where

$$a_0 = f(0) + \frac{1}{\pi}\int_0^{\pi} u \int_0^{\frac{\pi}{2}} f'(u \sin \theta) d\theta du,$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} u \cos ku \int_0^{\frac{\pi}{2}} f'(u \sin \theta)\, d\theta du.$$

9.171

$$f(x) = a_0 x^n + \sum_{k=1}^{\infty} a_k J_n(\alpha_k x) \qquad 0 < x < 1,$$

where

$$J_{n+1}(\alpha_k) = 0,$$

$$a_0 = 2(n+1)\int^1 f(x) x^{n+1}\, dx,$$

$$a_k = \frac{2}{[J_n(\alpha_k)]^2}\int_0^1 x f(x) J_n(\alpha_k x) dx.$$

(Bridgman, Phil. Mag. 16, p. 947, 1908)

9.172

$$f(x) = \sum_{k=1}^{\infty} A_k J_0(\mu_k x) \qquad a < x < b,$$

where:

$$a\frac{J_0'(\mu_k a)}{J_0(\mu_k a)} = p\mu_k + \frac{q}{\mu_k},$$

and

$$J_0(\mu_k b) = 0,$$

$$A_k = 2\,\frac{\int_a^b x f(x) J_0(\mu_k x) dx - p f(a) J_0(\mu_k a)}{b^2 J_0'^2(\mu_k b) - a^2 J_0'^2(\mu_k a) - (a^2 + 2p) J_0^2(\mu_k a)}.$$

(Stephenson, Phil. Mag. 14, p. 547, 1907)

SPECIAL EXPANSIONS IN BESSEL'S FUNCTIONS

9.180

1. $\sin x = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(x),$

2. $\cos x = J_0(x) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(x).$

9.181

1. $$\cos(x \sin\theta) = J_0(x) + 2\sum_{k=1}^{\infty} J_{2k}(x) \cos 2k\theta,$$

2. $$\sin(x \sin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(x) \sin(2k+1)\theta.$$

9.182

1. $$\left(\frac{x}{2}\right)^n = \sum_{k=0}^{\infty} \frac{(n+2k)\,(n+k-1)!}{k!} J_{n+2k}(x),$$

2. $$\sqrt{\frac{2x}{\pi}} = \sum_{k=0}^{\infty} \frac{(4k+1)\,(2k)!}{2^{2k}(k!)^2} J_{2k+\frac{1}{2}}(x).$$

9.183

$$\frac{dJ_\nu(x)}{d\nu} = \left\{\log\frac{x}{2} - \psi(\nu+1)\right\} J_\nu(x) + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\nu+2k}{k(\nu+k)} J_{\nu+2k}(x)$$

$$= J_\nu(x) \log\frac{x}{2} - \sum_{k=0}^{\infty} (-1)^k \frac{\psi(\nu+k+1)}{k!\;\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}. \qquad \text{(see 6.61)}$$

9.200 The differential equation:

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(\mu^2 - \frac{n(n+1)}{x^2}\right)y = 0$$

with the substitution:

$$z = y\sqrt{x}, \qquad \mu x = \rho$$

becomes:

$$\frac{d^2z}{d\rho^2} + \frac{1}{\rho}\frac{dz}{d\rho} + \left(1 - \frac{(n+\frac{1}{2})^2}{\rho^2}\right)z = 0$$

which is Bessel's equation of order $n + \dfrac{1}{2}$.

9.201 Two independent solutions are:

$$z = J_{n+\frac{1}{2}}(\rho),$$

$$z = J_{-n-\frac{1}{2}}(\rho).$$

9.202 Special values.

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x,$$

$$J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right),$$

$$J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{3}{x^2} - 1\right)\sin x - \frac{3}{x}\cos x\right\},$$

$$J_{\frac{7}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{15}{x^3} - \frac{6}{x}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x\right\},$$

$$J_{\frac{9}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\sin x - \left(\frac{105}{x^3} - \frac{10}{x}\right)\cos x\right\}.$$

9.203

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x,$$

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right),$$

$$J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3}{x}\sin x + \left(\frac{3}{x^2} - 1\right)\cos x\right\},$$

$$J_{-\frac{7}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left\{\left(\frac{15}{x^2} - 1\right)\sin x + \left(\frac{15}{x^3} - \frac{6}{x}\right)\cos x\right\},$$

$$J_{-\frac{9}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\left(\frac{105}{x^3} - \frac{10}{x}\right)\sin x + \left(\frac{105}{x^4} - \frac{45}{x^2} + 1\right)\cos x\right\}.$$

9.204

$$H^{\mathrm{I}}_{\frac{1}{2}}(x) = -i\sqrt{\frac{2}{\pi x}}e^{ix},$$

$$H^{\mathrm{I}}_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{ix}\left(1 + \frac{i}{x}\right),$$

$$H^{\mathrm{I}}_{\frac{5}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{ix}\left\{\frac{3}{x} + i\left(\frac{3}{x^2} - 1\right)\right\}.$$

9.205

$$H^{\mathrm{II}}_{\frac{1}{2}}(x) = i\sqrt{\frac{2}{\pi x}}e^{-ix},$$

$$H^{\mathrm{II}}_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left(1 - \frac{i}{x}\right),$$

$$H^{\mathrm{II}}_{\frac{5}{2}}(x) = -\sqrt{\frac{2}{\pi x}}e^{-ix}\left\{\frac{3}{x} - i\left(\frac{3}{x^2} - 1\right)\right\}.$$

9.210 The differential equation:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\left(1+\frac{\nu^2}{x^2}\right)y=0,$$

with the substitution,

$$x=iz,$$

becomes Bessel's equation.

9.211 Two independent solutions of **9.210** are:

$$I_\nu(x)=i^{-\nu}J_\nu(ix),$$

$$K^\nu(x)=e^{\frac{\nu+1}{2}\pi i}\frac{\pi}{2}H^1_\nu(ix).$$

9.212 If $\nu=n$, an integer:

$$I_n(x)=\sum_{k=0}^{\infty}\frac{1}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{n+2k},$$

$$K_n(x)=i^{n+1}\frac{\pi}{2}H^I_n(x).$$

9.213

$$I_\nu(x)=\frac{1}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu\int_0^\pi\cosh(x\cos\phi)\sin^{2\nu}\phi d\phi,$$

$$K_\nu(x)=\frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\left(\frac{x}{2}\right)^\nu\int_0^\infty\sinh^{2\nu}\phi e^{-x\cosh\phi}\,d\phi.$$

9.214 If x is large, to a first approximation:

$$I_n(x)=(2\pi x\cosh\beta)^{-\frac{1}{2}}e^{x(\cosh\beta-\beta\sinh\beta)},$$

$$K_n(x)=\pi(2\pi x\cosh\beta)^{-\frac{1}{2}}e^{-x(\cosh\beta-\beta\sinh\beta)},$$

$$n=x\sinh\beta.$$

9.215 Ber and Bei Functions.

$$\text{ber}\,x+i\,\text{bei}\,x=I\,(x\sqrt{i}),$$

$$\text{ber}\,x-i\,\text{bei}\,x=I_0(ix\sqrt{i}),$$

$$\text{ber}\,x=1-\frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4+\frac{1}{(4!)^2}\left(\frac{x}{2}\right)^8-\ldots$$

9.216 Ker and Kei Functions:

$$\operatorname{ker} x + i \operatorname{kei} x = K_0(x\sqrt{i}),$$

$$\operatorname{ker} x - i \operatorname{kei} x = K_0(ix\sqrt{i}),$$

$$\operatorname{ker} x = \left(\log\frac{2}{x} - \gamma\right)\operatorname{ber} x + \frac{\pi}{4}\operatorname{bei} x - \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right)\left(\frac{x}{2}\right)^4 + \frac{1}{(4!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right)\left(\frac{x}{2}\right)^8 - \cdots$$

$$\operatorname{kei} x = \left(\log\frac{2}{x} - \gamma\right)\operatorname{bei} x - \frac{\pi}{4}\operatorname{ber} x + \left(\frac{x}{2}\right)^2 - \frac{1}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\left(\frac{x}{2}\right)^6 + \cdots.$$

9.220 The Bessel-Clifford Differential Equation:

$$x\frac{d^2y}{dx^2} + (\nu + 1)\frac{dy}{dx} + y = 0.$$

With the substitution:

$$z = x^{\nu/2}y \qquad u = 2\sqrt{x},$$

the differential equation reduces to Bessel's equation.

9.221 Two independent solutions of **9.220** are:

$$C_\nu(x) = x^{-\frac{\nu}{2}} J_\nu(2\sqrt{x}) = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{k!\,\Gamma(\nu + k + 1)},$$

$$D_\nu(x) = x^{-\frac{\nu}{2}} Y_\nu(2\sqrt{x}).$$

9.222

$$C_{\nu+1}(x) = -\frac{d}{dx}C_\nu(x),$$

$$xC_{\nu+2}(x) = (\nu + 1)C_{\nu+1}(x) - C_\nu(x).$$

9.223 If $\nu = n$, an integer:

$$C_n(x) = (-1)^n \frac{d^n}{dx^n}C_0(x),$$

$$C_0(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{(k!)^2}$$

9.224 Changing the sign of ν, the corresponding solution of:

$$x\frac{d^2y}{dx^2} - (\nu - 1)\frac{dy}{dx} + y = 0,$$

$$y = x^\nu C_\nu(x).$$

9.225 If ν is half an odd integer:

$$C_{\frac{1}{2}}(x) = \frac{\sin(2\sqrt{x} + \epsilon)}{2\sqrt{x}},$$

$$C_{\frac{3}{2}}(x) = -\frac{d}{dx} C_{\frac{1}{2}}(x) = \frac{\sin(2\sqrt{x} + \epsilon)}{4x^{\frac{3}{2}}} - \frac{\cos(2\sqrt{x} + \epsilon)}{2x},$$

$$C_{\frac{5}{2}}(x) = -\frac{d}{dx} C_{\frac{3}{2}}(x) = \frac{3 - 4x}{8x^{\frac{5}{2}}} \sin(2\sqrt{x} + \epsilon) - \frac{3\cos(2\sqrt{x} + \epsilon)}{4x^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$C_{-\frac{1}{2}}(x) = -\cos(2\sqrt{x} + \epsilon),$$

$$C_{-\frac{3}{2}}(x) = x^{\frac{3}{2}} C_{\frac{3}{2}}(x),$$

$$C_{-\frac{5}{2}}(x) = x^{\frac{5}{2}} C_{\frac{5}{2}}(x).$$

$$\cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot$$

ϵ is arbitrary so as to give a second arbitrary constant.

9.226 For x negative, the solution of the equation:

$$x\frac{d^2y}{dx^2} + (2\nu + 1)\frac{dy}{dx} - y = 0,$$

when ν is half an odd integer, is obtained from the values in **9.225** by changing sin and cos to sinh and cosh respectively.

9.227

$$(m + n + 1)\int C_{m+1}(x)C_{n+1}(x)\,dx = -xC_{m+1}(x)C_{n+1}(x) - C_m(x)C_n(x),$$

$$(m + n + 1)\int x^{m+n}C_m(x)C_n(x)\,dx = x^{m+n+1}\left\{xC_{m+1}(x)C_{n+1}(x) + C_m(x)C_n(x)\right\}.$$

9.228

1. $$\int_0^\pi C_{-\frac{1}{2}}(x\cos^2\phi)\,d\phi = \pi C_0(x).$$

2. $$\int_0^\pi C_{\frac{1}{2}}(x\cos^2\phi)\,d\phi = \pi C_1(x).$$

3. $$\int_0^\pi C_0(x\sin^2\phi)\sin\phi\,d\phi = C_{\frac{1}{2}}(x).$$

4. $$\int_0^\pi C_1(x\sin^2\phi)\sin^3\phi\,d\phi = C_{\frac{5}{2}}(x).$$

5. $$\int_0^\pi C_1(x\sin^2\phi)\sin\phi\,d\phi = \frac{1 - \cos 2\sqrt{x}}{x}.$$

9.229 Many differential equations can be solved in a simpler form by the use of the C_n functions than by the use of Bessel's functions.

(Greenhill, Phil. Mag. 38, p. 501, 1919)

9.240 The differential equation:

$$\frac{d^2y}{dx^2} + \frac{2(n+1)}{x}\frac{dy}{dx} + y = 0,$$

with the change of variable:

$$y = zx^{-n-\frac{1}{2}},$$

becomes Bessel's equation **9.200**.

9.241 Solutions of **9.240** are:

1. $$y = x^{-n-\frac{1}{2}}\ J_{n+\frac{1}{2}}(x).$$
2. $$y = x^{-n-\frac{1}{2}}\ Y_{n+\frac{1}{2}}(x).$$
3. $$y = x^{-n-\frac{1}{2}}\ H^{\mathrm{I}}_{n+\frac{1}{2}}(x).$$
4. $$y = x^{-n-\frac{1}{2}}\ H^{\mathrm{II}}_{n+\frac{1}{2}}(x).$$

9.242 The change of variable:

$$x = 2\sqrt{z},$$

transforms equation **9.240** into the Bessel-Clifford differential equation **9.220**. This leads to a general solution of **9.240**:

$$y = C_{n+\frac{1}{2}}\left(\frac{x^2}{4}\right).$$

When n is an integer the equations of **9.225** may be employed.

$$C_{\frac{1}{2}}\left(\frac{x^2}{4}\right) = \frac{\sin(x+\epsilon)}{x},$$

$$C_{\frac{3}{2}}\left(\frac{x^2}{4}\right) = \frac{2\sin(x+\epsilon)}{x^3} - \frac{\cos(x+\epsilon)}{x},$$

$$\cdots$$

9.243 The solution of

$$\frac{d^2y}{dx^2} + \frac{2(n+1)}{x}\frac{dy}{dx} - y = 0,$$

may be obtained from **9.242** by writing sinh and cosh for sin and cos respectively.

9.244 The differential equation **9.240** is also satisfied by the two independent functions (when n is an integer):

$$\psi_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}$$

$$\Psi_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x}$$

$$= \frac{1\cdot3\cdot5\,.\,.\,.\,(2n-1)}{x^{2n+1}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{2^k k!(1-2n)(3-2n)\,.\,.\,.\,.\,(2k-2n-1)},$$

9.245 The general solution of **9.240** may be written:

$$y = \left(\frac{1}{x}\frac{d}{dx}\right)^n \frac{Ae^{ix}+Be^{-ix}}{x}.$$

9.246 Another particular solution of **9.240** is:

$$y = f_n(x) = \left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{-ix}}{x} = \Psi_n(x) - i\psi_n(x),$$

$$f_n(x) = \frac{i^n e^{-ix}}{x^{n+1}}\left\{1 + \frac{n(n+1)}{2ix} + \frac{(n-1)n(n+1)(n+2)}{2\cdot4\cdot(ix)^2} + \,.\,.\,.\,.\,.\, + \frac{1\cdot2\cdot3\,.\,.\,.\,.\,.\,2n}{2\cdot4\cdot6\,.\,.\,.\,.\,2n(ix)^n}\right\}.$$

9.247 The functions $\psi_n(x)$, $\Psi_n(x)$, $f_n(x)$ satisfy the same recurrence formulae:

$$\frac{d\psi_n(x)}{dx} = -x\psi_{n+1}(x),$$

$$x\frac{d\psi_n(x)}{dx} + (2n+1)\psi_n(x) = \psi_{n-1}(x).$$

9.260 The differential equation:

$$\frac{d^2y}{dx^2} - \frac{n(n+1)}{x^2}y + y = 0,$$

with the change of variable:

$$y = u\sqrt{x}$$

is transformed into Bessel's equation of order $n + \frac{1}{2}$.

9.261 Solutions of **9.260** are:

$$S_n(x) = \sqrt{\frac{\pi x}{2}}\,J_{n+\frac{1}{2}}(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\sin x}{x}.$$

$$C_n(x) = (-1)^n\sqrt{\frac{\pi x}{2}}\,J_{-n-\frac{1}{2}}(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{\cos x}{x}.$$

$$E_n(x) = C_n(x) - i\,S_n(x) = x^{n+1}\left(-\frac{1}{x}\frac{d}{dx}\right)^n \frac{e^{-ix}}{x}.$$

9.262 The functions $S_n(x)$, $C_n(x)$, $E_n(x)$ satisfy the same recurrence formulae:

1. $\frac{dS_n(x)}{dx} = \frac{n+1}{x}S_n(x) - S_{n+1}(x).$

2. $$\frac{dS_n(x)}{dx} = S_{n-1}(x) - \frac{n}{x} S_n(x).$$

3. $$S_{n+1}(x) = \frac{2n+1}{x} S_n(x) - S_{n-1}(x).$$

9.30 The hypergeometric differential equation:

$$x(1-x)\frac{d^2y}{dx^2} + \left\{ \gamma - (\alpha+\beta+1)x \right\} \frac{dy}{dx} - \alpha\beta y = 0.$$

9.31 The equation **9.30** is satisfied by the hypergeometric series:

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha}{1}\frac{\beta}{\gamma} x + \frac{\alpha(\alpha+1)}{1\cdot 2}\frac{\beta(\beta+1)}{\gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1\cdot 2\cdot 3}\frac{\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} x^3 + \dots\dots$$

The series converges absolutely when $x<1$ and diverges when $x>1$. When $x=+1$ it converges only when $\alpha+\beta-\gamma<0$, and then absolutely. When $x=-1$ it converges only when $\alpha+\beta-\gamma-1<0$, and absolutely if $\alpha+\beta-\gamma<0$.

9.32

$$\frac{d}{dx} F(\alpha, \beta, \gamma, x) = \frac{\alpha\beta}{\gamma} F(\alpha+1, \beta+1, \gamma+1, x).$$

$$F(\alpha, \beta, \gamma, 1) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}.$$

9.33 Representation of various functions by hypergeometric series.

$$(1+x)^n = F(-n, \beta, \beta, -x),$$

$$\log(1+x) = xF(1, 1, 2, -x),$$

$$e^x = \operatorname*{Limit}_{\beta=\infty} F\left(1, \beta, 1, \frac{x}{\beta}\right),$$

$$(1+x)^n + (1-x)^n = 2F\left(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{1}{2}, x^2\right),$$

$$\log\frac{1+x}{1-x} = 2xF\left(\frac{1}{2}, 1, \frac{3}{2}, x^2\right),$$

$$\cos nx = F\left(\frac{n}{2}, -\frac{n}{2}, \frac{1}{2}, \sin^2 x\right),$$

$$\sin nx = n\sin xF\left(\frac{n+1}{2}, \frac{1-n}{2}, \frac{3}{2}, \sin^2 x\right),$$

$$\cosh x = \underset{\alpha = \beta = \infty}{\text{Limit}} F\left(\alpha, \beta, \frac{1}{2}, \frac{x^2}{4\alpha\beta}\right),$$

$$\sin^{-1} x = xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right),$$

$$\tan^{-1} x = xF\left(\frac{1}{2}, 1, \frac{3}{2}, -x^2\right),$$

$$P_n(x) = F\left(-n, n+1, 1, \frac{1-x}{2}\right),$$

$$Q_n(x) = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}\frac{1}{x^{n+1}}F\left(\frac{n+1}{2}, \frac{n+2}{2}, n+\frac{3}{2}, \frac{1}{x^2}\right).$$

9.4 Heaviside's Operational Methods of Solving Partial Differential Equations.

9.41 The partial differential equation,

$$a\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t},$$

where a is a constant, may be solved by Heaviside's operational method.

Writing $\frac{\partial}{\partial t} = p$, and $\frac{p}{a} = q^2$, the equation becomes,

$$\frac{\partial^2 u}{\partial x^2} = q^2 u,$$

whose complete solution is $u = e^{qx}A + e^{-qx}B$, where A and B are integration constants to be determined by the boundary conditions. In many applications the solution $u = e^{-qx}B$, only, is required: and the boundary conditions will lead to $u = e^{-qx}f(q)u_0$, where u_0 is a constant. If $e^{-qx}f(q)$ be expanded in an infinite power series in q, and the integral and fractional, positive and negative powers of p be interpreted as in **9.42**, the resulting series will be a solution of the differential equation, satisfying the boundary conditions, and reducing to $u = 0$ at $t = 0$. The expansion of $e^{-qx}f(q)$ may be carried out in two or more ways, leading to series suitable for numerical calculation under different conditions.

9.42 Fractional Differentiation and Integration.

In the following expressions, 1 stands for a function of t which is zero up to $t = 0$, and equal to 1 for $t > 0$.

9.421

$$p^{\frac{1}{2}} 1 = \frac{1}{\sqrt{\pi t}}$$

$$p^{\frac{3}{2}} 1 = \frac{1}{2t\sqrt{\pi t}}$$

$$p^{\frac{5}{2}} 1 = \frac{3}{2^2 t^2 \sqrt{\pi t}}$$

. . .

. . .

$$p^{\frac{2n+1}{2}} 1 = (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n t^n \sqrt{\pi t}}$$

9.422

$$p\, 1 = 0$$
$$p^2 1 = 0$$
$$p^3 1 = 0$$

. . .

. . .

$$p^n 1 = 0$$

9.423

$$p^{-\frac{1}{2}} = 2\sqrt{\frac{t}{\pi}}$$

$$p^{-\frac{3}{2}} = \frac{2^2 t}{3}\sqrt{\frac{t}{\pi}}$$

$$p^{-\frac{5}{2}} = \frac{2^3 t^2}{3 \cdot 5}\sqrt{\frac{t}{\pi}}$$

. . .

. . .

$$p^{-\frac{2n+1}{2}} 1 = \frac{2^{2n-1} t^n}{1 \cdot 3 \cdot 5 \ldots (2n+1)}\sqrt{\frac{t}{\pi}}$$

9.424

$$\frac{1}{p^\nu} = \frac{t^\nu}{\Gamma(1+\nu)},$$

where ν may have any real value, except a negative integer. (Conjectu

9.425

$$\frac{p}{p-a} 1 = e^{at}$$

$$\frac{1}{p-a} 1 = \frac{1}{a}(e^{at} - 1)$$

9.426 With $p = aq^2$,

$$q^{2n+1} 1 = (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{(2at)^n \sqrt{\pi a t}}$$

$$q^{-2n} 1 = \frac{(at)^n}{n!}.$$

9.427

$$qe^{-qx}1 = \frac{1}{\sqrt{\pi a t}}e^{-\frac{x^2}{4at}}$$

9.428 If $z = \dfrac{x}{2\sqrt{at}}$,

$$e^{-qx} = \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-v^2}dv$$

$$\frac{1}{q}e^{-qx} = \frac{x}{\sqrt{\pi}}\int_z^{\infty} e^{-v^2}\frac{dv}{v^2}.$$

9.43 Many examples of the use of this method are given by Heaviside: Electromagnetic Theory, Vol. II. Bromwich, Proceedings Cambridge Philosophical Society, XX, p. 411, 1921, has justified its application by the method of contour integration and applied it to the solution of a problem in the conduction of heat.

9.431 Herlitz, Arkiv for Matematik, Astronomi och Fysik, XIV, 1919, has shown that the same methods may be applied to the more general partial differential equations of the type,

$$\sum_{\alpha,\beta} A_{\alpha,\beta}(x)\frac{\partial^{\alpha+\beta}(u)}{\partial x^{\alpha}\,\partial t^{\beta}} = 0,$$

and the relations of **9.42** are valid.

9.44 Heaviside's Expansion Theorem.

The operational solution of the differential equation of **9.41**, or the more general equation, **9.431**, satisfying the given boundary conditions, may be written in the form,

$$u = \frac{F(p)}{\Delta(p)}u_0,$$

where $F(p)$ and $\Delta(p)$ are known functions of $p = \dfrac{\partial}{\partial t}$. Then Heaviside's Expansion Theorem is:

$$u = u_0\left\{\frac{F(0)}{\Delta(0)} + \sum\frac{F(\alpha)}{\alpha\Delta'(\alpha)}e^{\alpha t}\right\},$$

where α is any root, except 0, of $\Delta(p) = 0$, $\Delta'(p)$ denotes the first derivative of $\Delta(p)$ with respect to p, and the summation is to be taken over all the roots of $\Delta(p) = 0$. This solution reduces to $u = 0$ at $t = 0$.

Many applications of this expansion theorem are given by Heaviside, Electromagnetic Theory, II, and III; Electrical Papers, Vol. II. Herlitz, **9.431**, has also applied this expansion theorem to the solution of the problem of the distribution of magnetic induction in cylinders and plates.

9.45 Bromwich's Expansion Theorem. Bromwich has extended Heaviside's Expansion Theorem as follows. If the operational solution of the partial differential equation of **9.41**, obtained to satisfy the boundary conditions, is

$$u = \frac{F(p)}{\Delta(p)}(Gt)$$

where G is a constant, then the solution of the differential equation is

$$u = G\left\{N_0 t + N_1 + \sum \frac{F(\alpha)}{\alpha^2 \Delta'(\alpha)} e^{\alpha t}\right\},$$

where N_0 and N_1 are defined by the expansion,

$$\frac{F(p)}{\Delta(p)} = N_0 + N_1 p + N_2 p^2 + \ldots;$$

α is any root of $\Delta(p) = 0$, $\Delta'(p)$ is the first derivative of $\Delta(p)$ with respect to p, and the summation is over all the roots, α. This solution reduces to $u = 0$ at $t = 0$. Phil. Mag. 37, p. 407, 1919; Proceedings London Mathematical Society, 15, p. 401, 1916.

9.9 References to Bessel Functions.

Nielsen: Handbuch der Theorie der Cylinder Funktionen.
Leipzig, 1904.

The notation and definitions given by Nielsen have been adopted in the present collection of formulae. The only difference is that Nielsen uses an upper index, $J^n(x)$, to denote the order, where the more usual custom of writing $J_n(x)$ is here employed. In place of $H_1{}^n$ and $H_2{}^n$ used by Nielsen for the cylinder functions of the third kind, $H_n{}^{\mathrm{I}}$ and $H_n{}^{\mathrm{II}}$ are employed in this collection.

Gray and Mathews: Treatise on Bessel Functions.
London, 1895.[1]

The Bessel Function of the second kind, $Y_n(x)$, employed by Gray and Mathews is the function

$$\frac{\pi}{2} Y_n(x) + (\log 2 - \gamma) J_n(x),$$

of Nielsen.

Schafheitlin: Die Theorie der Besselschen Funktionen.
Leipzig, 1908.

Schafheitlin defines the function of the second kind, $Y_n(x)$, in the same way as Nielsen, except that its sign is changed.

Note. A Treatise on the Theory of Bessel Functions, by G. N. Watson, Cambridge University Press, 1922, has been brought out while this volume is in press. This Treatise gives by far the most complete account of the theory and properties of Bessel Functions that exists, and should become the standard work on the subject with respect to notation. A particularly valuable feature is the Collection of Tables of Bessel Functions at the end of the volume and the Bibliography, giving references to all the important works on the subject.

9.91 Tables of Legendre, Bessel and allied functions.

$P_n(x)$ (9.001).

[1] A second edition of Gray and Mathews' Treatise, prepared by A. Gray and T. M. MacRobert, has been published (1922) while this volume is in press. The notation of the first edition has been altered in some respects.

B. A. Report, 1879, pp. 54–57. Integral values of n from 1 to 7; from $x = 0.01$ to $x = 1.00$, interval 0.01, 16 decimal places.

Jahnke and Emde: Funktionentafeln, p. 83; same to 4 decimal places.

$P_n(\cos\theta)$

Phil. Trans. Roy. Soc. London, 203, p. 100, 1904. Integral values of n from 1 to 20, from $\theta = 0$ to $\theta = 90$, interval 5, 7 decimal places.

Phil. Mag. 32, p. 512, 1891. Integral values of n from 1 to 7, $\theta = 0$ to $\theta = 90$, interval 1; 4 decimal places. Reproduced in Jahnke and Emde, p. 85.

Tallquist, Acta Soc. Sc. Fennicae, Helsingfors, 33, pp. 1–8. Integral values of n from 1 to 8; $\theta = 0$ to $\theta = 90$, interval 1, 10 decimal places.

Airey, Proc. Roy. Soc. London, 96, p. 1, 1919. Tables by means of which zonal harmonics of high order may be calculated.

Lodge, Phil. Trans. Roy. Soc. London, 203, 1904, p. 87. Integral values of n from 1 to 20; $\theta = 0$ to $\theta = 90$, interval 5, 7 decimal places. Reprinted in Rayleigh, Collected Works, Volume V, p. 162.

$\dfrac{\partial P_n(\cos\theta)}{\partial\theta}$.

Farr, Proc. Roy. Soc. London, 64, 199, 1899. Integral values of n from 1 to 7; $\theta = 0$ to $\theta = 90$, interval 1, 4 decimal places. Reproduced in Jahnke and Emde, p. 88.

$J_0(x)$, $J_1(x)$ (9.101).

Meissel's tables, $x = 0.01$ to $x = 15.50$, interval 0.01, to 12 decimal places, are given in Table I of Gray and Mathews' Treatise on Bessel's Functions.

Aldis, Proc. Roy. Soc. London 66, 40, 1900. $x = 0.1$ to $x = 6.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Funktionentafeln, Table III. $x = 0.01$ to $x = 15.50$, interval 0.01, 4 decimal places.

$J_n(x)$ (9.101).

Gray and Mathews, Table II. Integral values of n from $n = 0$ to $n = 60$; integral values of x from $x = 1$ to $x = 24$, 18 decimal places.

Jahnke and Emde, Table XXIII, same, to 4 significant figures.

B. A. Report, 1915, p. 29; $n = 0$ to $n = 13$.

$x = 0.2$ to $x = 6.0$	interval 0.2	6 decimal places,
$x = 6.0$ to $x = 16.0$	interval 0.5	10 decimal places.

Hague, Proc. London Physical Soc. 29, 211, 1916–17, gives graphs of $J_n(x)$ for integral values of n from 0 to 12, and $n = 18$, x ranging from 0 to 17.

$-\frac{\pi}{2}Y_0(x) = G_0(x)$; $-\frac{\pi}{2}Y_1(x) = G_1(x)$.

B. A. Report, 1913, pp. 116–130. $x = 0.01$ to $x = 16.0$, interval 0.01, 7 decimal places.

B. A. Report, 1915, $x = 6.5$ to $x = 15.5$, interval 0.5, 10 decimal places.

Aldis, Proc. Roy. Soc. London, 66, 40, 1900: $x = 0.1$ to $x = 6.0$. Interval 0.1, 21 decimal places.

Jahnke and Emde, Tables VII and VIII, functions denoted $K_0(x)$ and $K_1(x)$, $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 0.01$ to $x = 0.99$, interval 0.01; $x = 1.0$ to $x = 10.3$, interval 0.1; 4 decimal places.

$-\frac{\pi}{2} Y_n(x) = G_n(x).$

B. A. Report, 1914, p. 83. Integral values of n from 0 to 13. $x = 0.01$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 16.0$, interval 0.5; 5 decimal places.

$\frac{\pi}{2} Y_0(x) + (\log 2 - \gamma)J_0(x),$ Denoted $Y_0(x)$ and $Y_1(x)$

$\frac{\pi}{2} Y_1(x) + (\log 2 - \gamma)J_1(x).$ respectively in the tables.

B. A. Report, 1914, p. 76, $x = 0.02$ to $x = 15.50$, interval 0.02, 6 decimal places.

B. A. Report, 1915, p. 33, $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 15.5$, interval 0.5, 10 decimal places.

Jahnke and Emde, Table VI, $x = 0.01$ to $x = 1.00$, interval 0.01; $x = 1.0$ to $x = 10.2$, interval 0.1, 4 decimal places.

$Y_0(x)$, $Y_1(x)$. Denoted $N_0(x)$ and $N_1(x)$ respectively.

Jahnke and Emde, Table IX, $x = 0.1$ to $x = 10.2$, interval 0.1, 4 decimal places.

$\frac{\pi}{2} Y_n(x) + (\log 2 - \gamma) J_n(x).$ Denoted $Y_n(x)$ in tables.

B. A. Report, 1915. Integral values of n from 1 to 13. $x = 0.2$ to $x = 6.0$, interval 0.2; $x = 6.0$ to $x = 15.5$, interval 0.5, 6 decimal places.

$J_{n+\frac{1}{2}}(x)$.

Jahnke and Emde, Table II. Integral values of n from $n = 0$ to $n = 6$, and $n = -1$ to $n = -7$; $x = 0$ to $x = 50$, interval 1.0, 4 figures.

$J_{\frac{1}{3}}(x)$, $J_{-\frac{1}{3}}(x)$.

Watson, Proc. Roy. Soc. London, 94, 204, 1918.

$x = 0.05$ to $x = 2.00$ interval 0.05,

$x = 2.0$ to $x = 8.0$ interval 0.2,

4 decimal places.

$J_\alpha(\alpha)$, $J_{\alpha-1}(\alpha)$

$-\frac{\pi}{2}Y_\alpha(\alpha)$, $-\frac{\pi}{2} Y_{\alpha-1}(\alpha)$. Denoted $G_\alpha(\alpha)$ and $G_{\alpha-1}(\alpha)$ respectively.

$\frac{\pi}{2} Y_\alpha(\alpha) + (\log 2 - \gamma)J_\alpha(\alpha),$

$\frac{\pi}{2} Y_{\alpha-1}(\alpha) + (\log 2 - \gamma)J_{\alpha-1}(\alpha).$ Denoted $-V_\alpha(\alpha)$ and $-V_{\alpha-1}(\alpha)$.

Tables of these six functions are given in the B. A. Report, 1916, as follows:

From α	to α	Interval
1	50	1
50	100	5
100	200	10
200	400	20
400	1000	50
1000	2000	100
2000	5000	500
5000	20000	1000
20000	30000	10000
100,000		
500,000		
1,000,000		

$I_0(x)$, $I_1(x)$ (9.211).

Aldis, Proc. Roy. Soc. London, 64, pp. 218–223, 1899; $x = 0.1$ to $x = 6.0$, interval 0.1; $x = 6.0$ to $x = 11.0$, interval 1.0, 21 decimal places.

Jahnke and Emde, Tables XI and XII, 4 places:

$x = 0.01$ to $x = 5.10$	interval 0.01,
$x = 5.10$ to $x = 6.0$	interval 0.1,
$x = 6.0$ to $x = 11.0$	interval 1.0.

$\mathrm{I}_0(x)$ (9.211).

B. A. Report, 1896; $x = 0.001$ to $x = 5.100$, interval 0.001, 9 decimal places.

$\mathrm{I}_1(x)$ (9.211).

B. A. Report, 1893; $x = 0.001$ to $x = 5.100$, interval 0.001, 9 decimal places.

Gray and Mathews, Table V, $x = 0.01$ to $x = 5.10$, interval 0.01, 9 decimal places.

$\mathrm{I}_n(x)$ (9.211).

B. A. Report, 1889, pp. 28–32; integral values of n from 0 to 11, $x = 0.2$ to $x = 6.0$, interval 0.2, 12 decimal places. These tables are reproduced in Gray and Mathews, Table VI.

Jahnke and Emde, Table XXIV; same ranges, to 4 places.

$J_0(x\sqrt{i}) \quad = X - iY,$

$\sqrt{2}J_1(x\sqrt{i}) \quad = X_1 + iY_1$

Aldis, Proc. Roy. Soc. London, 66, 142, 1900; $x = 0.1$ to $x = 6.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Tables XV and XVI, same range, to 4 places.

$J_0(x\sqrt{i})$.

Gray and Mathews, Table IV; $x = 0.2$ to $x = 6.0$, interval 0.2, 9 decimal places.

$Y_0(x\sqrt{i})$ **(9.104)** Denoted $N_0(x\sqrt{i})$ in table.

$H_0^1(x\sqrt{i})$, $H_1^1(x\sqrt{i})$.

Jahnke and Emde, Tables XVII and XVIII; $x = 0.2$ to $x = 6.0$, interval 0.2, 4-7 figures.

$$\frac{i\pi}{2} H_0^1(ix) = K_0(x),$$
$$-\frac{\pi}{2} H_0^1(ix) = K_1(x),$$
(9.212).

Aldis, Proc. Roy. Soc. London, 64, 219-223, 1899; $x = 0.1$ to $x = 12.0$, interval 0.1, 21 decimal places.

Jahnke and Emde, Table XIV; same, to 4 places.

$iH_0^1(ix)$, $-H_0^1(ix)$ **(9.107).**

Jahnke and Emde, Table XIII; $x = 0.12$ to $x = 6.0$, interval 0.2, 4 figures.

$\mathrm{ber}\, x$, $\mathrm{ber}'\, x$,
$\mathrm{bei}\, x$, $\mathrm{bei}'\, x$, **(9.215).**

B. A. Report, 1912; $x = 0.1$ to $x = 10.0$, interval 0.1, 9 decimal places.

Jahnke and Emde, Table XX; $x = 0.5$ to $x = 6.0$, interval 0.5, and $x = 8$, 10, 15, 20, 4 decimal places.

$\mathrm{ker}\, x$, $\mathrm{ker}'\, x$,
$\mathrm{kei}\, x$, $\mathrm{kei}'\, x$, **(9.216).**

B. A. Report, 1915; $x = 0.1$ to $x = 10.0$, interval 0.1, 7-10 decimal places

$\mathrm{ber}^2 x + \mathrm{bei}^2 x$,
$\mathrm{ber}'^2 x + \mathrm{bei}'^2 x$,
$\mathrm{ber}\, x\, \mathrm{bei}'\, x - \mathrm{bei}\, x\, \mathrm{ber}'\, x$,
$\mathrm{ber}\, x\, \mathrm{ber}'\, x + \mathrm{bei}\, x\, \mathrm{bei}'\, x$,
and the corresponding ker and kei functions.

B. A. Report, 1916; $x = 0.2$ to $x = 10.0$, interval 0.2, decimal places.

$S_n(x)$, $S'_n(x)$, $\log S_n(x)$, $\log S'_n(x)$,
$C_n(x)$, $C'_n(x)$, $\log C_n(x)$, $\log C'_n(x)$, **(9.261).**
$E_n(x)$, $E'_n(x)$, $\log E_n(x)$, $\log E'_n(x)$,

B. A. Report, 1916; integral values of n from 0 to 10, $x = 1.1$ to $x = 1.9$, interval 0.1, 7 decimal places.

$$G(x) = -\sqrt{2}\,\Pi\left(\frac{1}{4}\right) x^{\frac{1}{4}} J_{\frac{1}{4}}\left(\frac{x}{2}\right) = -\frac{1}{0.78012}\, x^{\frac{1}{4}} J_{\frac{1}{4}}\left(\frac{x}{2}\right)$$

$$D(x) = \frac{1}{\sqrt{2}}\,\Pi\left(-\frac{1}{4}\right) x^{\frac{1}{4}} J_{-\frac{1}{4}}\left(\frac{x}{2}\right) = \frac{1}{1.15407}\, x^{\frac{1}{4}} J_{-\frac{1}{4}}\left(\frac{x}{2}\right)$$

Table I of Jahnke and Emde gives these two functions to 3 decimal places for $x = 0.2$ to $x = 8.0$, interval 0.2, and $x = 8.0$ to $x = 12.0$, interval 1.0.

Roots of $J_0(x) = 0$.

Airey, Phil. Mag. 36, p. 241, 1918: First 40 roots (ρ) with corresponding values of $J_1(\rho)$, 7 decimal places.

Jahnke and Emde, Table IV, same, to 4 decimal places.

Roots of $J_1(x) = 0$.

Gray and Mathews, Table III, first 50 roots, with corresponding values of $J_0(x)$, 16 decimal places.

Airey, Phil. Mag. 36, p. 241: First 40 roots (r) with corresponding values of $J_0(r)$, 7 decimal places.

Jahnke and Emde, Table IV, same, to 4 decimal places.

Roots of $J_n(x) = 0$.

B. A. Report, 1917, first 10 roots, to 6 figures, for the following integral values of n: 0–10, 15, 20, 30, 40, 50, 75, 100, 200, 300, 400, 500, 750, 1000.

Jahnke and Emde, Table XXII, first 9 roots, 3 decimal places, integral values of n 0–9.

Roots of:

$(\log 2 - \gamma)J_n(x) + \frac{\pi}{2} Y_n(x) = 0.$ Denoted $Y_n(x) = 0$ in table.

Airey: Proc. London Phys. Soc. 23, p. 219, 1910–11. First 40 roots for $n = 0, 1, 2$, 5 decimal places.

Jahnke and Emde, Table X, first 4 roots for $n = 0, 1$. 4 decimal places.

Roots of:

$Y_0(x) = 0,$

$Y_1(x) = 0.$ Denoted $N_0(x)$ and $N_1(x)$ in tables.

Airey: l. c. First 10 roots, 5 decimal places.

Roots of:

$J_0(x) \pm (\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $J_0(x) \pm Y_0(x) = 0.$

$J_1(x) + (\log 2 - \gamma)J_1(x) + \frac{\pi}{2} Y_1(x) = 0.$ Denoted $J_1(x) + Y_1(x) = 0.$

$J_0(x) - 2(\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $J_0(x) - 2Y_0(x) = 0.$

$10J_0(x) \pm (\log 2 - \gamma)J_0(x) + \frac{\pi}{2} Y_0(x) = 0.$ Denoted $10J_0(x) \pm Y_0(x) = 0.$

Airey, l. c. First 10 roots, 5 decimal places.

Roots of:

$$\frac{J_{n+1}(x)}{J_n(x)} + \frac{I_{n+1}(x)}{I_n(x)} = 0.$$

Airey, l. c. First 10 roots: $n = 0$, 4 decimal places, $n = 1, 2, 3$, 3 decimal places.

Jahnke and Emde, Table XXV, first 5 roots for $n = 0$, 3 for $n = 1$, 2 for $n = 2$: 4 figures.

Airey, l. c. gives roots of some other equations involving Bessel's functions connected with the vibration of circular plates.

Roots of:

$$J_\nu(x)\,Y_\nu(x) = J_\nu(kx)\,Y_\nu(kx).$$

Jahnke and Emde, Table XXVI, first 6 roots, 4 decimal places, for $\nu = 0, 1/2, 1, 3/2, 2, 5/2$: $k = 1.2, 1.5, 2.0$.

Table XXVIII, first root, multiplied by $(k - 1)$ for $k = 1, 1.2, 1.5, 2-11, 19, 39, \infty$: ν same as above.

Table XXIX, first 4 roots, multiplied by $(k - 1)$ for certain irrational values of k, and $\nu = 0, 1$.

X. NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

By F. R. Moulton, Ph.D.,
Professor of Astronomy, University of Chicago;
Research Associate of the Carnegie Institution of Washington.

INTRODUCTION

Differential equations are usually first encountered in the final chapter of a book on integral calculus. The methods which are there given for solving them are essentially the same as those employed in the calculus. Similar methods are used in the first special work on the subject. That is, numerous types of differential equations are given in which the variables can be separated by suitable devices; little or nothing is said about the existence of solutions of other types, or about methods of finding the solutions. The false impression is often left that only exceptionally can differential equations be solved. Whatever satisfaction there may be in learning that some problems in geometry and physics lead to standard forms of differential equations is more than counterbalanced by the discovery that most practical problems do not lead to such forms.

10.01 The point of view adopted here and the methods which are developed can be best understood by considering first some simpler and better known mathematical theories. Suppose

1. $$F(x) = x^n + a_1 x^{n-1} + \ldots\ldots + a_{n-1}x + a_n = 0$$

is a polynomial equation in x having real coefficients $a_1, a_2, \ldots, a_n$. If n is 1, 2, 3, or 4 the values of x which satisfy the equation can be expressed as explicit functions of the coefficients. If n is greater than 4, formulas for the solution can not in general be written down. Nevertheless, it is possible to prove that n solutions exist and that at least one of them is real if n is odd. If the coefficients are given numbers, there are straightforward, though somewhat laborious, methods of finding the solutions. That is, even though general formulas for the solutions are not known, yet it is possible both to prove the existence of the solutions and also to find them in any special numerical case.

10.02 Consider as another illustration the definite integral

1. $$I = \int_a^b f(x)\, dx,$$

where $f(x)$ is continuous for $a \leqslant x \leqslant b$. If $F(x)$ is such a function that

2. $$\frac{dF}{dx} = f(x),$$

then $I = F(b) - F(a)$. But suppose no $F(x)$ can be found satisfying (2). It is nevertheless possible to prove that the integral I exists, and if the value of $f(x)$ is given for every value of x in the interval $a \leqq x \leqq b$, it is possible to find the numerical value of I with any desired degree of approximation. That is, it is not necessary that the primitive of the integrand of a definite integral be known in order to prove the existence of the integral, or even to find its value in any particular example.

10.03 The facts are analogous in the case of differential equations. Those having numerical coefficients and prescribed initial conditions can be solved regardless of whether or not their variables can be separated. They need to satisfy only mild conditions which are always fulfilled in physical problems. It is with a sense of relief that one finds he can solve, numerically, any particular problem which can be expressed in terms of differential equations.

10.04 This chapter will contain an account of a method of solving ordinary differential equations which is applicable to a broad class including all those which arise in physical problems. A large amount of experience has shown that the method is very convenient in practice. It must be understood that there is for it an underlying logical basis, involving refinements of modern analysis, which fully justifies the procedure. In other words, it can be proved that the process is capable of furnishing the solution with any desired degree of accuracy. The proofs of these facts belong to the domain of pure analysis and will not be given here.

10.10 Simpson's Method of Computing Definite Integrals. The method of solving differential equations which will be given later involves the computation of definite integrals by a special process which will be developed in this and the following sections.

Let t be the variable of integration, and consider the definite integral

1. $$F = \int_a^b f(t)\,dt.$$

This integral can be interpreted as the area between the t-axis and the curve $y = f(t)$ and bounded by the ordinates $t = a$ and $t = b$, figure 1.

Let $t_0 = a$, $t_n = b$, $y_i = f(t_i)$, and divide the interval $a \leqq t \leqq b$ up into n equal parts, each of length $h = (b - a)/n$. Then an approximate value of F is

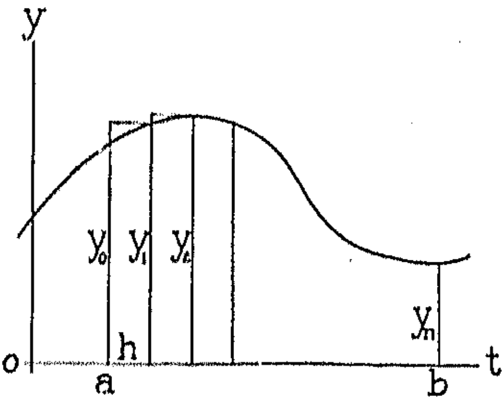

FIG. 1

2. $$F_0 = h(y_1 + y_2 + \dots + y_n).$$

This is the sum of rectangles whose ordinates, figure 1, are $y_1, y_2, \dots, y_n$.

10.11 A more nearly exact value can be obtained for the first two intervals,

y_0, y_1, y_2, and finding the area between the t-axis and this curve and bounded by the ordinates t_0 and t_2. The equation of the curve is

1. $$y = a_0 + a_1(t - t_0) + a_2(t - t_0)^2,$$

where the coefficients a_0, a_1, and a_2 are determined by the conditions that y shall equal y_0, y_1, and y_2 at t equal to t_0, t_1 and t_2 respectively; or

2. $$\begin{cases} y_0 = a_0, \\ y_1 = a_0 + a_1(t_1 - t_0) + a_2(t_1 - t_0)^2, \\ y_2 = a_0 + a_1(t_2 - t_0) + a_2(t_2 - t_0)^2. \end{cases}$$

It follows from these equations and $t_2 - t_1 = t_1 - t_0 = h$ that

3. $$\begin{cases} a_0 = y_0, \\ a_1 = -\dfrac{1}{2h}(3y_0 - 4y_1 + y_2), \\ a_2 = \dfrac{1}{2h^2}(y_0 - 2y_1 + y_2). \end{cases}$$

The definite integral $\int_{t_0}^{t_2} y\,dt$ is approximately

$$I = \int_{t_0}^{t_2}\left[a_0 + a_1(t - t_0) + a_2(t - t_0)^2\right] dt = 2h\left[a_0 + a_1 h + \frac{4}{3}a_2 h^2\right],$$

which becomes as a consequence of (3)

4. $$I = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

10.12 The value of the integral over the next two intervals, or from t_2 to t_4, can be computed in the same way. If n is even, the approximate value of the integral from t_0 to t_n is therefore

$$F_1 = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots\cdots + 4y_{n-1} + y_n\right].$$

This formula, which is due to Simpson, gives results which are usually remarkably accurate considering the simplicity of the arithmetical operations.

10.13 If a curve of the third degree had been passed through the four points y_0, y_1, y_2, and y_3, the integral corresponding to (4), but over the first three intervals, would have been found to be

$$I = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right].$$

10.20 Digression on Difference Functions. For later work it will be necessary to have some properties of the successive differences of the values of a function for equally spaced values of its argument.

$$\begin{aligned}
\Delta_1 y_1 &= y_1 - y_0, \\
\Delta_1 y_2 &= y_2 - y_1, \\
&\cdots\cdots \\
\Delta_1 y_n &= y_n - y_{n-1}, \\
&\cdots\cdots\cdots
\end{aligned}$$

These are the first differences of the values of the function y for successive values of t. All the successive intervals for t are supposed to be equal.

10.21 In a similar way the second differences are defined by

$$\begin{aligned}
\Delta_2 y_2 &= \Delta_1 y_2 - \Delta_1 y_1, \\
\Delta_2 y_3 &= \Delta_1 y_3 - \Delta_1 y_2, \\
&\cdots\cdots\cdots \\
\Delta_2 y_n &= \Delta_1 y_n - \Delta_1 y_{n-1}, \\
&\cdots\cdots\cdots
\end{aligned}$$

10.22 In a similar way third differences are defined by

$$\begin{aligned}
\Delta_3 y_3 &= \Delta_2 y_3 - \Delta_2 y_2, \\
\Delta_3 y_4 &= \Delta_2 y_4 - \Delta_2 y_3, \\
&\cdots\cdots \\
\Delta_3 y_n &= \Delta_2 y_n - \Delta_2 y_{n-1}, \\
&\cdots\cdots\cdots
\end{aligned}$$

and obviously the process can be repeated as many times as may be desired.

10.23 The table of successive differences can be formed conveniently from the tabular values of the function and can be arranged in a table as follows:

TABLE 1

y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$
y_0			
y_1	$\Delta_1 y_1$		
y_2	$\Delta_1 y_2$	$\Delta_2 y_2$	
y_3	$\Delta_1 y_3$	$\Delta_2 y_3$	$\Delta_3 y_3$
....	....	....	....

In this table the numbers in each column are subtracted from those immediately below them and the remainders are placed in the next column to the right on the same line as the minuends. Variations from this precise arrangement could be, and indeed often have been, adopted.

10.24 A very important advantage of a table of differences is that it is almost sure to reveal any errors that may have been committed in computing the y_i. If a single y_i has an error ϵ, it follows from **10.20** that the first difference $\Delta_1 y_i$ will contain the error $+\epsilon$ and $\Delta_1 y_{i+1}$ will contain the error $-\epsilon$. But the seco[nd] differences $\Delta_2 y_i$, $\Delta_2 y_{i+1}$, and $\Delta_2 y_{i+2}$ will contain the respective errors $+\epsilon$, $+\epsilon$. Similarly, the third differences $\Delta_3 y_i$, $\Delta_3 y_{i+1}$, $\Delta_3 y_{i+2}$, and $\Delta_3 y_{i+3}$ will c[ontain] the respective errors $+\epsilon$, -3ϵ, $+3\epsilon$, $-\epsilon$. An error in a single y_i affects $j+1$ differences of order j, and the coefficients of the error are the binomial coeffi-

numbers in the various difference columns are zero. Now in such functions as ordinarily occur in practice the numerical values of the differences, if the intervals are not too great, decrease with rapidity and run smoothly. If an error is present, however, the differences of higher order become very irregular.

10.25 As an illustration, consider the function $y = \sin t$ for t equal to 10°, 15°, ... The following table gives the function and its successive differences, expressed in terms of units of the fourth decimal:[1]

TABLE II

t	$\sin t$	$\Delta_1 \sin t$	$\Delta_2 \sin t$	$\Delta_3 \sin t$
10°	1736			
15	2588	852		
20	3420	832	−20	
25	4226	806	−26	−6
30	5000	774	−32	−6
35	5736	736	−38	−6
40	6428	692	−44	−6
45	7071	643	−49	−5
50	7660	589	−54	−5
55	8191	531	−58	−4
60	8660	469	−62	−4
65	9063	403	−66	−4
70	9397	334	−69	−3

Suppose, however, that an error of two units had been made in determining the sine of 45° and that 7073 had been taken in place of 7071. Then the part of the table adjacent to this number would have been the following:

TABLE III

t	$\sin t$	$\Delta_1 \sin$	$\Delta_2 \sin t$	$\Delta_3 \sin t$
25°	4226			
30	5000	774		
35	5736	736	−38	
40	6428	692	−44	−6
45	7073	645	−47	−3
50	7660	587	−58	−11
55	8191	531	−56	+2
60	8660	469	−62	−6
65	9063	403	−66	−4

The irregularity in the numbers of the last column shows the existence of an error, and, in fact, indicates its location. In the third differences four numbers

[1] Often it is not necessary to carry along the decimal and zeros to the left of the first significant figure.

will be affected by an error in the value of the function. The erroneous numbers in the last column are clearly the second, third, fourth, and fifth. The algebraic sum of these four numbers equals the sum of the four correct numbers, or -18. Their average is -4.5. Hence the central numbers are probably -5 and -4. Since the errors in these numbers are -3ϵ and $+3\epsilon$, it follows that ϵ is probably $+2$. The errors in the second and fifth numbers are $+\epsilon$ and $-\epsilon$ respectively. On making these corrections and working back to the first column, it is found that 7073 should be replaced by 7071.

10.30 Computation of Definite Integrals by Use of Difference Functions.

Suppose the values of $f(t)$ are known for $t = t_{n-2}, t_{n-1}, t_n$, and t_{n+1}. Suppose it is desired to find the integral

1. $$I_n = \int_{t_n}^{t_{n+1}} f(t)\,dt.$$

The coefficients b_0, b_1, b_2, and b_3 of the polynomial can be determined, as above, so that the function

2. $$y = b_0 + b_1(t - t_n) + b_2(t - t_n)^2 + b_3(t - t_n)^3$$

shall take the same values as $f(t)$ for $t = t_{n-2}, t_{n-1}, t_n$, and t_{n+1}.

With this approximation to the function $f(t)$, the integral becomes (since $t_{n+1} - t_n = h$)

3. $$\begin{aligned} I_n &= \int_{t_n}^{t_{n+1}} \left[b_0 + b_1(t - t_n) + b_2(t - t_n)^2 + b_3(t - t_n)^3\right] dt \\ &= h\left[b_0 + \frac{1}{2}\, b_1 h + \frac{1}{3}\, b_2 h^2 + \frac{1}{4}\, b_3 h^3\right]. \end{aligned}$$

The coefficients b_0, b_1, b_2, and b_3 will now be expressed in terms of y_{n+1}, $\Delta_1 y_{n+1}$, $\Delta_2 y_{n+1}$, and $\Delta_3 y_{n+1}$. It follows from (2) that

4. $$\begin{cases} y_{n-2} = b_0 - 2b_1 h + 4b_2 h^2 - 8b_3 h^3, \\ y_{n-1} = b_0 - b_1 h + b_2 h^2 - b_3 h^3, \\ y_n \phantom{{}_{-1}} = b_0, \\ y_{n+1} = b_0 + b_1 h + b_2 h^2 + b_3 h^3. \end{cases}$$

Then it follows from the rules for determining the difference functions that

5. $$\begin{cases} \Delta_1 y_{n-1} = b_1 h - 3b_2 h^2 + 7b_3 h^3, \\ \Delta_1 y_n \phantom{{}_{-1}} = b_1 h - b_2 h^2 + b_3 h^3, \\ \Delta_1 y_{n+1} = b_1 h + b_2 h^2 + b_3 h^3. \end{cases}$$

6. $$\begin{cases} \Delta_2 y_n \phantom{{}_{+1}} = 2b_2 h^2 - 6b_3 h^3, \\ \Delta_2 y_{n+1} = 2b_2 h^2. \end{cases}$$

7. $$\Delta_3 y_{n+1} = 6b_3 h^3.$$

It follows from the last equations of these four sets of equations that

$$8. \qquad \begin{cases} b_0 &= y_{n+1} - \Delta_1 y_{n+1}, \\[2ex] b_1 h &= \Delta_1 y_{n+1} - \dfrac{1}{2}\Delta_2 y_{n+1} - \dfrac{1}{6}\Delta_3 y_{n+1}, \\[2ex] b_2 h^2 &= \dfrac{1}{2}\Delta_2 y_{n+1}, \\[2ex] b_3 h^3 &= \dfrac{1}{6}\Delta_3 y_{n+1}. \end{cases}$$

Therefore the integral (3) becomes

$$9. \qquad I_n = h\left[y_{n+1} - \frac{1}{2}\Delta_1 y_{n+1} - \frac{1}{12}\Delta_2 y_{n+1} - \frac{1}{24}\Delta_3 y_{n+1} - \cdots\cdots\right].$$

The coefficients of the higher order terms $\Delta_4 y_{n+1}$ and $\Delta_5 y_{n+1}$ are $-\frac{19}{720}$ and $\cancel{-\frac{3}{160}}$ respectively.

10.31 Obviously, if it were desired, the integral from t_{n-2} to t_{n-1}, or over any other part of this interval, could be computed by the same methods. For example, the integral from t_{n-1} to t_n is

$$\begin{aligned} I_{n-1} &= \int_{t_{n-1}}^{t_n} f(t)\,dt, \\ &= h\left[y_{n+1} - \frac{3}{2}\Delta_1 y_{n+1} + \frac{5}{12}\Delta_2 y_{n+1} + \frac{1}{24}\Delta_3 y_{n+1} + \cdots\cdots\right]. \end{aligned}$$

NUMERICAL ILLUSTRATIONS

10.32 Consider first the application of Simpson's method. Suppose it is required to find

$$I = \int_{25^\circ}^{55^\circ} \sin t\,dt = -\left|\cos t\right|_{25^\circ}^{55^\circ} = 0.3327.$$

On applying **10.12** with the numbers taken from Table I, it is found that

$$I_1 = \frac{5^\circ}{3}\left[.4226 + 2.0000 + 1.1472 + 2.5712 + 1.4142 + 3.0640 + .8191\right],$$

which becomes, on reducing 5° to radians,

$$I_1 = 0.3327,$$

agreeing to four places with the correct result.

10.33 On applying **10.11** (4) and omitting alternate entries in Table II, it is found that

$$I = \int_{25^\circ}^{45^\circ} \sin t\,dt = \frac{10^\circ}{3}\left[.4226 + 2.2944 + .7071\right] = 0.1992,$$

which is also correct to four places. These formulas could hardly be surpassed in ease and convenience of application.

10.34 Now consider the application of **10.30** (9). As it stands it furnishes the integral over the single interval t_n to t_{n+1}. If it is desired to find the integral from t_n to t_{n+m}, the formula for doing so is obviously the sum of m formulas such as (9), the value of the subscript going from $n+1$ to $n+m+1$, or

$$I_{n,m} = h\Big[\Big(y_{n+1} + \cdots\cdots + y_{n+m+1}\Big) - \frac{1}{2}\Big(\Delta_1 y_{n+1} + \cdots\cdots + \Delta_1 y_{n+m+1}\Big)$$
$$- \frac{1}{12}\Big(\Delta_2 y_{n+1} + \cdots\cdots + \Delta_2 y_{n+m+1}\Big) - \frac{1}{24}\Big(\Delta_3 y_{n+1} + \cdots\cdots + \Delta_3 y_{n+m+1}\Big) + \cdots\Big].$$

On applying this formula to the numbers of Table I, it is found that

$$\begin{aligned} I = \int_{25°}^{55°} \sin t\, dt = 5°[&(.5000 + .5736 + .6428 + .7071 + .7660 + .8191) \\ &- \frac{1}{2}(.0774 + .0736 + .0692 + .0643 + .0589 + .0531) \\ &+ \frac{1}{12}(.0032 + .0038 + .0044 + .0049 + .0054 + .0058) \\ &+ \frac{1}{24}(.0006 + .0006 + .0006 + .0005 + .0005 + .0004)] \\ = 0.3427,& \end{aligned}$$

agreeing to four places with the exact value. When a table of differences is at hand covering the desired range this method involves the simplest numerical operations. It must be noted, however, that some of the required differences necessitate a knowledge of the value of the function for earlier values of the argument than the lower limit of the integral.

10.40 Reduced Form of the Differential Equations. Differential equations which arise from physical problems usually involve second derivatives. For example, the differential equation satisfied by the motion of a vibrating tuning fork has the form

$$\frac{d^2x}{dt^2} = -kx,$$

where k is a constant depending on the tuning fork.

10.41 The differential equations for the motion of a body subject to gravity and a retardation which is proportional to its velocity are

$$\begin{cases} \dfrac{d^2x}{dt^2} = -c\dfrac{dx}{dt}, \\[2ex] \dfrac{d^2y}{dt^2} = -c\dfrac{dy}{dt} - g, \end{cases}$$

where c is a constant depending on the resisting medium and the mass and shape of the body, while g is the acceleration of gravity.

10.42 The differential equations for the motion of a body moving subject to the law of gravitation are

$$\begin{cases} \dfrac{d^2x}{dt^2} = -k^2\dfrac{x}{r^3}, \\ \dfrac{d^2y}{dt^2} = -k^2\dfrac{y}{r^3}, \\ \dfrac{d^2z}{dt^2} = -k^2\dfrac{z}{r^3}, \\ r^2 = x^2 + y^2 + z^2. \end{cases}$$

10.43 These examples illustrate sufficiently the types of differential equations which arise in practical problems. The number of the equations depends on the problem and may be small or great. In the problem of three bodies there are nine equations. The equations are usually not independent as is illustrated in **10.42,** where each equation involves all three variables x, y, and z through r. On the other hand, equations **10.41** are mutually independent for the first does not involve y or its derivatives and the second does not involve x or its derivatives. The right members may involve x, y, and z as is the case in **10.42,** or they may involve the first derivatives, as is the case in **10.41,** or they may involve both the coördinates and their first derivatives. In some problems they also involve the independent variable t.

10.44 Hence physical problems usually lead to differential equations which are included in the form

$$\begin{cases} \dfrac{d^2x}{dt^2} = f\left(x, y, \dfrac{dx}{dt}, \dfrac{dy}{dt}, t\right), \\ \dfrac{d^2y}{dt^2} = g\left(x, y, \dfrac{dx}{dt}, \dfrac{dy}{dt}, t\right), \end{cases}$$

where f and g are functions of the indicated arguments. Of course, the number of equations may be greater than two.

10.45 If we let

$$x' = \frac{dx}{dt}, \quad y' = \frac{dy}{dt},$$

equations **10.44** can be written in the form

$$\begin{cases} \dfrac{dx}{dt} = x', \\ \dfrac{dx'}{dt} = f(x, y, x', y', t), \\ \dfrac{dy}{dt} = y', \\ \dfrac{dy'}{dt} = g(x, y, x', y', t). \end{cases}$$

10.46 If we let $x = x_1$, $x' = x_2$, $y = x_3$, $y' = x_4$, ... equations **10.45** are included in the form

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(x_1, x_2, \ldots, x_n, t), \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots \\ \dfrac{dx_n}{dt} = f_n(x_1, x_2, \ldots, x_n, t). \end{cases}$$

This is the final standard form to which it will be supposed the differential equations are reduced.

10.50 Definition of a Solution of Differential Equations. For simplicity in writing, suppose the differential equations are two in number and write them in the form

$$\text{1.}\qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = g(x, y, t), \end{cases}$$

where f and g are known functions of their arguments. Suppose $x = a$, $y = b$ at $t = 0$. Then

$$\text{2.}\qquad \begin{cases} x = \phi(t), \\ y = \psi(t), \end{cases}$$

is the solution of (1) satisfying these initial conditions if ϕ and ψ are such functions that

$$\text{3.}\qquad \begin{aligned} \phi(0) &= a, \\ \psi(0) &= b, \\ \frac{d\phi}{dt} &= f(\phi, \psi, t), \\ \frac{d\psi}{dt} &= g(\phi, \psi, t), \end{aligned}$$

the last two equations being satisfied for all $0 \leqq t \leqq T$, where T is a positive constant, the largest value of t for which the solution is determined. It is not necessary that ϕ and ψ be given by any formulas — it is sufficient that they have the properties defined by (3). Solutions *always exist*, though it will not be proved here, *if f and g are continuous functions of t and have derivatives with respect to both x and y.*

10.51 Geometrical Interpretation of a Solution of Differential Equations. Geometrical interpretations of definite integrals have been of great value not

practical means of obtaining their numerical values. The same things are true in the case of differential equations.

For simplicity in the geometrical representation, consider a single equation

1. $$\frac{dx}{dt} = f(x, t),$$

where $x = a$ at $t = 0$. Suppose the solution is

2. $$x = \phi(t).$$

Equation (2) defines a curve whose coördinates are x and t. Suppose it is represented by figure 2. The value of the tangent to the curve at every point on it is given by equation (1), for there is, corresponding to each point, a pair of values of x and t which gives $\frac{dx}{dt}$, the value of the tangent, when substituted in the right member of equation (1).

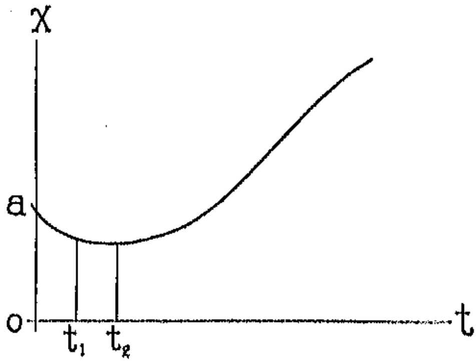

FIG. 2

Consider the initial point on the curve, viz. $x = a$, $t = 0$. The tangent at this point is $f(a, 0)$. The curve lies close to the tangent for a short distance from the initial point. Hence an approximate value of x at $t = t_1$, t_1 being small, is the ordinate of the point where the tangent at a intersects the line $t = t_1$, or

$$x_1 = f(a, 0)t_1.$$

The tangent at x_1, t_1 is defined by (1), and a new step in the solution can be made in the same way. Obviously the process can be continued as long as x and t have values for which the right member of (1) is defined. And the same process can be applied when there are any number of equations. While the steps of this process can be taken so short that it will give the solution with any desired degree of accuracy, it is not the most convenient process that may be employed. It is the one, however, which makes clearest to the intuitions the nature of the solution.

10.6 Outline of the Method of Solution. Consider equations **10.50** (1) and their solution (2). The problem is to find functions ϕ and ψ having the properties (2). If we integrate the last two equations of **10.50** (3) we shall have

$$\begin{cases} \phi = a + \int_0^t f(\phi, \psi, t)\, dt, \\ \psi = b + \int_0^t g(\phi, \psi, t)\, dt. \end{cases}$$

The difficulty arises from the fact that ϕ and ψ are not known in advance and the integrals on the right can not be formed. Since ϕ and ψ are the solution values of x and y, we may replace them by the latter in order to preserve the original notation, and we have

2. $$\begin{cases} x = a + \int_0^t f(x, y, t)\, dt, \\ y = b + \int_0^t g(x, y, t)\, dt. \end{cases}$$

If x and y do not change rapidly in numerical value, then $f(x, y, t)$ and $g(x, y, t)$ will not in general change rapidly, and a first approximation to the values of x and y satisfying equations (2) is

3. $$\begin{cases} x_1 = a + \int_0^t f(a, b, t)\, dt, \\ y_1 = b + \int_0^t g(a, b, t)\, dt, \end{cases}$$

at least for values of t near zero. Since a and b are constants, the integrands in (3) are known and the integrals can be computed. If the primitives can not be found the integrals can be computed by the methods of **10.1** or **10.3**.

After a first approximation has been found a second approximation is given by

4. $$\begin{cases} x_2 = a + \int_0^t f(x_1, y_1, t)\, dt, \\ y_2 = b + \int_0^t g(x_1, y_1, t)\, dt. \end{cases}$$

The integrands are again known functions of t because x_1 and y_1 were determined as functions of t by equations (3). Consequently x_2 and y_2 can be computed. The process can evidently be repeated as many times as is desired. The nth approximation is

5. $$\begin{cases} x_n = a + \int_0^t f(x_{n-1}, y_{n-1}, t)\, dt, \\ y_n = b + \int_0^t g(x_{n-1}, y_{n-1}, t)\, dt. \end{cases}$$

There is no difficulty in carrying out the process, but the question arises whether it converges to the solution. The answer, first established by Picard, is that, as n increases, x_n and y_n tend toward the solution for all values of t for which all the approximations belong to those values of x, y, and t for which f and g have the properties of continuity with respect to t and differentiability with respect to x and y. If, for example, $f = \dfrac{\sin x}{x^2}$ and the value of x_n tends towards zero for $t = T$, then the solution can not be extended beyond $t = T$.

It is found in practice that the longer the interval over which the integration is extended in the successive approximations, the greater the number of approximations which must be made in order to obtain a given degree of accuracy. In fact, it is preferable to take first a relatively short interval and to find the solution over this interval with the required accuracy, and then to continue from the end values of this interval over a new interval. This is what is done in actual work. The details of the most convenient methods of doing it will be explained in the succeeding sections.

$\Delta_3 x_{n-3}$, $\Delta_3 x_{n-2}$, $\Delta_3 x_{n-1}$, and $\Delta_3 x_n$ vary. For example, in Table II it is easy to see that $\Delta_3 \sin 75^\circ$ is almost certainly -3. It follows from **10.20, 1, 2** that

$$3. \quad \begin{cases} \Delta_2 x_{n+1} = \Delta_3 x_{n+1} + \Delta_2 x_n, \\ \Delta_1 x_{n+1} = \Delta_2 x_{n+1} + \Delta_1 x_n, \\ x_{n+1} = \Delta_1 x_{n+1} + x_n. \end{cases}$$

After the adopted value of $\Delta_3 x_{n+1}$ has been written in its column the successive entries to the left can be written down by simple additions to the respective numbers on the line of t_n. For example, it is found from Table II that $\Delta_2 \sin 75^\circ = -72$, $\Delta_1 \sin 75^\circ = 262$, $\sin 75^\circ = 9659$. This is, indeed, the correct value of $\sin 75^\circ$ to four places.

Now having extrapolated approximate values of x_{n+1} and y_{n+1} it remains to compute f and g for $x = x_{n+1}$, $y = y_{n+1}$, $t = t_{n+1}$. The next step is to pass curves through the values of f and g for $t = t_{n+1}, t_n, t_{n-1}, \ldots$ and to compute the integrals (2). This is the precise problem that was solved in **10.30**, the only difference being that in that section the integrand was designated by y. On applying equation **10.30** (9) to the computation of the integrals (2), the latter give

$$4. \quad \begin{cases} x_{n+1} = x_n + h\left[f_{n+1} - \frac{1}{2}\Delta_1 f_{n+1} - \frac{1}{12}\Delta_2 f_{n+1} - \frac{1}{24}\Delta_3 f_{n+1} \ldots\right], \\ y_{n+1} = y_n + h\left[g_{n+1} - \frac{1}{2}\Delta_1 g_{n+1} - \frac{1}{12}\Delta_2 g_{n+1} - \frac{1}{24}\Delta_3 g_{n+1} \ldots\right], \end{cases}$$

where

$$5. \quad \begin{cases} f_{n+1} = f(x_{n+1}, y_{n+1}, t_{n+1}), \\ g_{n+1} = g(x_{n+1}, y_{n+1}, t_{n+1}). \end{cases}$$

The right members of (4) are known and therefore x_{n+1} and y_{n+1} are determined.

It will be recalled that f_{n+1} and g_{n+1} were computed from extrapolated values of x_{n+1} and y_{n+1}, and hence are subject to some error. They should now be recomputed with the values of x_{n+1} and y_{n+1} furnished by (4). Then more nearly correct values of the entire right members of (4) are at hand and the values of x_{n+1} and y_{n+1} should be corrected if necessary. If the interval h is small it will not generally be necessary to correct x_{n+1} and y_{n+1}. But if they require corrections, then new values of f_{n+1} and g_{n+1} should be computed. In practice it is advisable to take the interval h so small that one correction to f_{n+1} and g_{n+1} is sufficient.

After x_{n+1} and y_{n+1} have been obtained, values of x and y at t_{n+2} can be found in precisely the same manner, and the process can be continued to $t = t_{n+3}, t_{n+4}, \ldots$. If the higher differences become large and irregular it is advisable to interpolate values at the mid-intervals of the last two steps and to continue with an interval half as great. On the other hand, if the higher differences become very small it is advisable to proceed with an interval twice as great as that used in the earlier part of the computation.

The foregoing, expressed in words, seems rather complicated. As a matter of fact, it goes very simply in practice, as will be shown in section **10.9**.

10.8 The Start of the Construction of the Solution. Suppose the differential equations are again

$$\text{1.}\qquad \begin{cases} \dfrac{dx}{dt} = f(x, y, t), \\ \dfrac{dy}{dt} = g(x, y, t), \end{cases}$$

with the initial conditions $x = a$, $y = b$ at $t = 0$. Only the initial values of x and y are known. But it follows from (1) that the rates of change of x and y at $t = 0$ are $f(a, b, 0)$ and $g(a, b, 0)$ respectively. Consequently, first approximations to values of x and y at $t = t_1 = h$ are

$$\text{2.}\qquad \begin{cases} x_1^{(1)} = a + hf(a, b, 0), \\ y_1^{(1)} = b + hg(a, b, 0). \end{cases}$$

Now it follows from (1) that the rates of change of x and y at $x = x_1$, $y = y_1$, $t = t_1$ are approximately $f(x_1^{(1)}, y_1^{(1)}, t_1)$ and $g(x_1^{(1)}, y_1^{(1)}, t_1)$. These rates will be different from those at the beginning, and the average rates of change for the first interval will be nearly the average of the rates at the beginning and at the end of the interval. Therefore closer approximations than those given in (2) to the values of x and y at $t = t_1$ are

$$\text{3.}\qquad \begin{cases} x_1^{(2)} = a + \tfrac{1}{2}h\left[f(a, b, 0) + f(x_1^{(1)}, y_1^{(1)}, t_1)\right], \\ y_1^{(2)} = b + \tfrac{1}{2}h\left[g(a, b, 0) + g(x_1^{(1)}, y_1^{(1)}, t_1)\right]. \end{cases}$$

The process could be repeated on the first interval, but it is not advisable when the interval is taken as short as it should be.

The rates of change at the beginning of the second interval are approximately $f(x_1^{(2)}, y_1^{(2)}, t_1)$ and $g(x_1^{(2)}, y_1^{(2)}, t_1)$ respectively. Consequently, first approximations to the values of x and y at $t = t_2$, where $t_2 - t_1 = h$, are

$$\text{4.}\qquad \begin{cases} x_2^{(1)} = x_1^{(2)} + hf(x_1^{(2)}, y_1^{(2)}, t_1), \\ y_2^{(1)} = y_1^{(2)} + hg(x_1^{(2)}, y_1^{(2)}, t_1). \end{cases}$$

With these values of x and y approximate values of f_2 and g_2 are computed. Since f_0, g_0; f_1, g_1 are known, it follows that $\Delta_1 f_0$, $\Delta_1 g_0$; $\Delta_1 f_1$, and $\Delta_1 g_1$ are also known. Hence equations (4) of **10.7**, for $n + 1 = 2$, can be used, with the exception of the last terms in the right members, for the computation of x_2 and y_2.

At this stage of work $x_0 = a$, $y_0 = b$; x_1, y_1; x_2, y_2 are known, the first pair exactly and the last two pairs with considerable approximation. After f_2 and g_2 have been computed, x_1 and y_1 can be corrected by **10.31** for $n = 1$. Then approximate values of x_3 and y_3 can be extrapolated by the method explained in the preceding section, after which approximate values of f_3 and g_3 can be computed. With these values and the corresponding difference functions, x_2 and y_2 can be corrected by using **10.31**. Then after correcting all the corresponding differences of all the functions, the solution is fully started and proceeds by the method given in the preceding section.

10.9 Numerical Illustration. In this section a numerical problem will be treated which will illustrate both the steps which must be taken and also the method of

arranging the work. A convenient arrangement of the computation which preserves a complete record of all the numerical work is very important.

Suppose the differential equation is

$$\text{1.}\qquad \begin{cases} \dfrac{d^2x}{dt^2} = -(1+\kappa^2)x + 2\kappa^2x^3, \\[2ex] x = 0,\ \dfrac{dx}{dt} = 1 \text{ at } t = 0. \end{cases}$$

The problem of the motion of a simple pendulum takes this form when expressed in suitable variables. This problem is chosen here because it has an actual physical interpretation, because it can be integrated otherwise so as to express t in terms of x, and because it will illustrate sufficiently the processes which have been explained.

Equation (1) will first be integrated so as to express t in terms of x. On multiplying both sides of (1) by $2\dfrac{dx}{dt}$ and integrating, it is found that the integral which satisfies the initial conditions is

$$\text{2.}\qquad \left(\frac{dx}{dt}\right)^2 = (1-x^2)\,(1-\kappa^2x^2).$$

On separating the variables this equation gives

$$\text{3.}\qquad t = \int_0^x \frac{dx}{\sqrt{(1-x^2)\,(1-\kappa^2x^2)}}.$$

Suppose $\kappa^2 < 1$ and that the upper limit x does not exceed unity. Then

$$\text{4.}\qquad \frac{1}{\sqrt{1-\kappa^2x^2}} = 1 + \frac{1}{2}\kappa^2x^2 + \frac{3}{8}\kappa^4x^4 + \frac{5}{16}\kappa^6x^6 + \dots,$$

where the right member is a converging series. On substituting (4) into (3) and integrating, it is found that

$$\text{5.}\quad t = \sin^{-1}x + \tfrac{1}{4}\Big[-x\sqrt{1-x^2} + \sin^{-1}x\Big]\kappa^2 + \tfrac{3}{8}\Big[-x^3\sqrt{1-x^2} - \tfrac{3}{4}x(1-x^2)^{\frac{3}{2}}$$
$$+ \tfrac{3}{8}x\sqrt{1-x^2} + \tfrac{3}{8}\sin^{-1}x\Big]\kappa^4 + \dots\dots\Big].$$

When $x = 1$ this integral becomes

$$\text{6.}\qquad T = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2\kappa^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\kappa^6 + \dots\right].$$

Equation (5) gives t for any value of x between -1 and $+1$. But the problem is to determine x in terms of t. Of course, if a table is constructed giving t for many values of x, it may be used inversely to obtain the value of x corresponding to any value of t. The labor involved is very great. When κ^2 is given numerically it is simpler to compute the integral (3) by the method of **10.1** or **10.3**.

In mathematical terms, t is an elliptical integral of x of the first kind, and the inverse function, that is, x as a function of t, is the sine-amplitude function, which has the real period $4T$.

Suppose $\kappa^2 = \frac{1}{2}$ and let $y = \frac{dx}{dt}$. Then equation (1) is equivalent to the two equations

7. $$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -\dfrac{3}{2}x + x^3, \end{cases}$$

which are of the form **10.50** (1), where

8. $$\begin{cases} f = y, \\ g = -\dfrac{3}{2}x + x^3, \end{cases}$$

and $x = 0$, $y = 1$ at $t = 0$.

The first step is to determine the interval which is to be used in the start of the solution. No general rule can be given. The larger f_0 and g_0 the smaller must the interval be taken. A fairly good rule is in general to take h so small that hf_0 and hg_0 shall not be greater than 1000 times the permissible error in the results. In the present instance we may take $h = 0.1$.

First approximations to x and y at $t = 0.1$ are found from the initial conditions and equations **10.8** (2) to be

9. $$\begin{cases} x_1^{(1)} = 0 + \dfrac{1}{10}1 = 0.1000, \\ y_1^{(1)} = 1 + \dfrac{1}{10}0 = 1.0000. \end{cases}$$

It follows from (8) and these values of x_1 and y_1 that

10. $$\begin{cases} f(x_1^{(1)}, y_1^{(1)}, t_1) = 1.0000, \\ g(x_1^{(1)}, y_1^{(1)}, t_1) = -0.1490. \end{cases}$$

Hence the more nearly correct values of x_1 and y_1, which are given by **10.8** (3), are

11. $$\begin{cases} x_1^{(2)} = 0 + \dfrac{0.1}{2}[1.0000 + 1.0000] = 0.1000, \\ y_1^{(2)} = 1 + \dfrac{0.1}{2}[0.0000 - 0.1490] = 0.9925. \end{cases}$$

Since in this particular problem $x = \int y\,dt$, it is not necessary to compute both f and g by the exact process explained in section **10.8**, for after y has been determined x is given by the integral. It follows from (7), (8), (10), and (11) that a first approximation to the value of y at $t = t_2 = 0.2$ is

12. $$y_2^{(1)} = .9925 - \frac{1}{10}.1490 = .9776.$$

With the values of y at 0, .1, .2 given by the initial conditions and in equations (9) and (12), the first trial y-table is constructed as follows:

First Trial y-Table

t	y	$\Delta_1 y$	$\Delta_2 y$
0	.1000		
.1	.0925	$-.0075$	
.2	.0770	$-.0140$	$-.0074$

Since $y = f$ it now follows from the first equations of (11) and **10.7** (4) for $n = 1$ that an approximate value of x_2 is

13. $$x_2^{(0)} = 0.1000 + \frac{1}{10}\left[.0770 + \frac{1}{2}.0140 + \frac{1}{12}.0074\right] = .1986.$$

With this value of x_2 it is found from the second of (8) that $g_2 = .2901$. Then the first trial g-table constructed from the values of g at $t = 0, 0.1, 0.2$, is:

First Trial g-Table

t	g	$\Delta_1 g$	$\Delta_2 g$
0	.0000		
.1	$-.1400$	$-.1400$	
.2	$-.2901$	$-.1411$	$+.0079$

Then the second equation of **10.7** (4) gives for $n = 1$ the more nearly correct value of y_2,

14. $$y_2 = .0925 + \frac{1}{10}\left[-.2901 + \frac{1}{2}.1411 - \frac{1}{12}.0079\right] = .0705.$$

This value of y_2 should replace the last entry in the first trial y-table. When this is done it is found that $\Delta_1 y_2 = -.0220$, $\Delta_2 y_2 = -.0145$. Then the first equation of **10.7** (4) gives

15. $$x_2 = .1000 + \frac{1}{10}\left[.0705 + \frac{1}{2}.0220 + \frac{1}{12}.0145\right] = .1983.$$

The computation is now well started although x_1, y_1, x_2, and y_2 are still subject to slight errors. The values of x_1 and y_1 can be corrected by applying **10.31** for $n = 1$. It is necessary first to compute a more nearly correct value of g_2 by using the value of x_2 given in (15). The result is $g_2 = -.2896$, $\Delta_1 g_2 = -.1406$, $\Delta_2 g_2 = +.0084$. Then the second equation of **10.7** (4) gives

16. $$y_2 = .0925 + \frac{1}{10}\left[-.2896 + \frac{1}{2}.1406 - \frac{1}{12}.0084\right] = .0705,$$

agreeing with (14). This value of y_2 is therefore essentially correct. An application of **10.31** then gives

after which it is found that $g_1 = -.1486$, $\Delta_1 g_1 = -.1486$. Now the first trial y-table can be corrected by using the value of y_2 given in (14). The result is:

Second Trial y-Table

t	y	$\Delta_1 y$	$\Delta_2 y$
0	1.0000		
.1	.9925	$-.0075$	
.2	.9705	$-.0220$	$-.0145$

In order to correct x_2 and y_2 by the same method, which is the most convenient one to follow, it is necessary first to obtain approximate values of g_3 and y_3. The trial g-table can be corrected by computing g with the values of x given by (17) and (15). Then the line for g_3 can be extrapolated. The results are:

Second Trial g-Table

t	g	$\Delta_1 g$	$\Delta_2 g$
0	.0000		
.1	$-.1486$	$-.1486$	
.2	$-.2896$	$-.1410$	$+.0076$
.3	$-.4230$	$-.1334$	$+.0076$

Then the second equation of **10.7** (4) gives for $n = 2$,

18. $$y_3 = .9705 + \frac{1}{10}\left[-.4230 + \frac{1}{2}.1334 - \frac{1}{12}.0076\right] = .9348.$$

When this is added to the second trial y-table, it is found that

19. $$y_3 = .9348,\quad \Delta_1 y_3 = -.0357,\quad \Delta_2 y_3 = -.0137,\quad \Delta_3 y_3 = +.0008.$$

Now x_2 and y_2 can be corrected by applying **10.31** to these numbers and those in the last line of the second trial g-table. The results are

20. $$\begin{cases} x_2 = .0997 + \dfrac{1}{10}\left[.9348 + \dfrac{3}{2}.0357 - \dfrac{5}{12}.0137 + \dfrac{1}{24}.0008\right] = .1980, \\ y_2 = .9925 + \dfrac{1}{10}\left[-.4230 + \dfrac{3}{2}.1334 + \dfrac{5}{12}.0076\right] = .9705. \end{cases}$$

The preliminary work is finished and x and y have been determined for $t = 0$, .1, and .2 with an error of probably not more than one unit in the last place. As the process is read over it may seem somewhat complicated, but this is largely because on the printed page preliminary values of the unknown quantities can

first steps are very simple and can be carried out in practice in a few minutes if the chosen time-interval is not too great.

The problem now reduces to simple routine. There are an x-table, a y-table (which in this problem serves also as an f-table), a g-table, and a schedule for computing g. It is advisable to use large sheets so that all the computations except the schedule for computing g can be kept side by side on the same sheet. The process consists of six steps: (1) Extrapolate a value of g_{n+1} and its differences in the g-table; (2) compute y_{n+1} by the second equation of 10.7 (4); (3) enter the result in the y-table and write down the differences; (4) use these results to compute x_{n+1} by the first equation of 10.7 (4); (5) with this value of x_{n+1} compute g_{n+1} by the g-computation schedule; and (6) correct the extrapolated value of g_{n+1} in the g-table.

Usually the correction to g_{n+1} will not be great enough to require a sensible correction to y_{n+1}. But if a correction is required, it should, of course, be made. It follows from the integration formulas 10.7 (4) and the way that the difference functions are formed that an error ϵ in g_{n+1} produces the error $\frac{3}{8}h\epsilon$ in y_{n+1}, and the corresponding error in x_{n+1} is $\frac{9}{64}h^2\epsilon$. It is never advisable to use so large a value of h that the error in x_{n+1} is appreciable. On the other hand, if the differences in the g-table and the y-table become so small that the second differences are insensible the interval may be doubled.

The following tables show the results of the computations in this problem reduced from five to four places.

Final x-Table

t	x	$\Delta_1 x$	$\Delta_2 x$	$\Delta_3 x$
0	.0000			
.1	.0997	.0997		
.2	.1980	.0983	−.0014	
.3	.2934	.0954	−.0029	−.0015
.4	.3847	.0913	−.0041	−.0012
.5	.4708	.0861	−.0052	−.0011
.6	.5508	.0800	−.0061	−.0009
.7	.6243	.0735	−.0065	−.0004
.8	.6909	.0666	−.0069	−.0004
.9	.7505	.0596	−.0070	−.0001
1.0	.8030	.0525	−.0071	−.0001
1.1	.8486	.0456	−.0069	+.0002
1.2	.8877	.0391	−.0065	+.0004
1.3	.9205	.0328	−.0063	+.0002
1.4	.9472	.0267	−.0061	+.0002
1.5	.9682	.0210	−.0057	+.0004
1.6	.9837	.0155	−.0055	+.0002
1.7	.9940	.0103	−.0052	+.0003
1.8	.9993	.0053	−.0050	+.0002
1.9	.9995	.0002	−.0051	−.0001

Final y-Table

t	y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$
0	1.0000			
.1	.9925	−.0075		
.2	.9705	−.0220	−.0145	
.3	.9352	−.0353	−.0133	+.0012
.4	.8882	−.0470	−.0117	+.0016
.5	.8320	−.0562	−.0092	+.0025
.6	.7687	−.0633	−.0071	+.0019
.7	.7009	−.0678	−.0045	+.0016
.8	.6308	−.0701	−.0023	+.0022
.9	.5602	−.0706	−.0005	+.0008
1.0	.4906	−.0696	+.0010	+.0015
1.1	.4231	−.0675	+.0021	+.0011
1.2	.3584	−.0647	+.0028	+.0007
1.3	.2968	−.0616	+.0031	+.0003
1.4	.2382	−.0586	+.0030	−.0001
1.5	.1824	−.0558	+.0028	−.0002
1.6	.1290	−.0534	+.0024	−.0004
1.7	.0775	−.0515	+.0019	−.0005
1.8	.0271	−.0504	+.0011	−.0008
1.9	−.0230	−.0501	+.0003	−.0008

Final g-Schedule

t	.1	.2	.3	.4	.5	.6	.7	.8	.9
$\log x$	8.9989	9.2967	9.4675	9.5851	9.6728	9.7410	9.7954	9.8394	9.8754
$\log x^3$	6.9967	7.8901	8.4025	8.7553	9.0184	9.2230	9.3862	9.5182	9.6259
$3x$	.2992	.5941	.8802	1.1541	1.4124	1.6524	1.8720	2.0747	2.2515
$-\frac{3}{2}x$	−.1496	−.2970	−.4401	−.5770	−.7062	−.8262	−.9365	−1.0364	−1.1257
x^3	.0010	.0077	.0252	.0569	.1044	.1671	.2430	.3298	.4227
u	−.1486	−.2893	−.4149	−.5201	−.6018	−.6591	−.6931	−.7066	−.7030

Final g-Table

t	g	$\Delta_1 g$	$\Delta_2 g$	$\Delta_3 g$
0	.0000			
.1	−.1486	−.1486		
.2	−.2893	−.1407	+.0079	
.3	−.4149	−.1256	+.0151	+.0072
.4	−.5201	−.1052	+.0204	+.0053
.5	−.6018	−.0817	+.0235	+.0031
.6	−.6591	−.0573	+.0244	+.0009
.7	−.6931	−.0340	+.0233	−.0011
.8	−.7066	−.0135	+.0205	−.0028
.9	−.7030	+.0036	+.0171	−.0034
1.0	−.6867	+.0163	+.0127	−.0044
1.1	−.6618	+.0249	+.0086	−.0041
1.2	−.6320	+.0298	+.0049	−.0037
1.3	−.6008	+.0312	+.0014	−.0035
1.4	−.5710	+.0298	−.0014	−.0028
1.5	−.5447	+.0263	−.0035	−.0021
1.6	−.5236	+.0211	−.0052	−.0017
1.7	−.5088	+.0148	−.0063	−.0011
1.8	−.5011	+.0077	−.0071	−.0008
1.9	−.5008	+.0003	−.0074	−.0003

Final g Schedule — *Continued*

1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
9.9047	9.9287	9.9483	9.9640	9.9764	9.9860	9.9929	9.9974	9.9997	9.9998
9.7141	9.7861	9.8449	9.8920	9.9292	9.9580	9.9787	9.9922	9.9991	9.9994
2.4090	2.5458	2.6631	2.7615	2.8416	2.9046	2.9511	2.9820	2.9979	2.9985
−1.2045	−1.2729	−1.3316	−1.3807	−1.4208	−1.4523	−1.4756	−1.4910	−1.4989	−1.4992
.5178	.6111	.6996	.7799	.8498	.9076	.9520	.9822	.9978	.9984
− .6867	− .6618	− .6320	− .6008	− .5710	− .5447	− .5236	− .5088	− .5011	− .5008

As has been remarked, large sheets should be used so that the x, y, and g-tables can be put side by side on one sheet. Then the t-column need be written but once for these three tables. The g-schedule, which is of a different type, should be on a separate sheet.

The differential equation (1) has an integral which becomes for $\kappa^2 = \frac{1}{2}$ and $\frac{dx}{dt} = y$.

21. $$y^2 + \frac{3}{2}x^2 - \frac{1}{4}x^4 = 1,$$

and which may be used to check the computation because it must be satisfied at every step. It is found on trial that (21) is satisfied to within one unit in the fourth place by the results given in the foregoing tables for every value of t.

The value of t for which $x = 1$ and $y = 0$ is given by (6). When $\kappa^2 = \frac{1}{2}$ it is found that $T = 1.8541$. It is found from the final x-table by interpolation based on first and second differences that x rises to its maximum unity for almost exactly this value of t; and, similarly, that y vanishes for this value of t.

XI ELLIPTIC FUNCTIONS

By Sir George Greenhill, F. R. S.

INTRODUCTION TO THE TABLES OF ELLIPTIC FUNCTIONS

By Sir George Greenhill

In the integral calculus, $\int \frac{dx}{\sqrt{X}}$, and more generally, $\int \frac{M + N\sqrt{X}}{P + Q\sqrt{X}}\,dx$, where M, N, P, Q are rational algebraical functions of x, can always be expressed by the elementary functions of analysis, the algebraical, circular, logarithmic or hyperbolic, so long as the degree of X does not exceed the second. But when X is of the third or fourth degree, new functions are required, called elliptic functions, because encountered first in the attempt at the rectification of an ellipse by means of an integral.

To express an elliptic integral numerically, when required in an actual question of geometry, mechanics, or physics and electricity, the integral must be normalised to a standard form invented by Legendre before the Tables can be employed; and these Tables of the Elliptic Functions have been calculated as an extension of the usual tables of the logarithmic and circular functions of trigonometry. The reduction to a standard form of any assigned elliptic integral that arises is carried out in the procedure described in detail in a treatise on the elliptic functions.

11.1. Legendre's Standard Elliptic Integral of the First Kind (E. I. I) is

$$F\phi = \int_0^\phi \frac{d\phi}{\sqrt{1 - \kappa^2 \sin^2\phi}} = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - \kappa^2 x^2)}} = u,$$

defining ϕ as the amplitude of u, to the modulus κ, with the notation,

$$\phi = \text{am } u$$
$$x = \sin\phi = \text{sin am } u$$

abbreviated by Gudermann to,

$$x = \text{sn } u$$
$$\cos\phi = \text{cn } u$$
$$\Delta\phi = \sqrt{(1 - \kappa^2\sin^2\phi)} = \Delta \text{ am } u = \text{dn } u,$$

and sn u, cn u, dn u are the three elliptic functions. Their differentiations are,

$$\frac{d\phi}{du} = \Delta\phi \qquad \text{or} \qquad \frac{d \text{ am } u}{du} = \text{dn } u$$

$$\frac{d\sin\phi}{du} = \cos\phi\cdot\Delta\phi \qquad \text{or} \qquad \frac{d \text{ sn } u}{du} = \text{cn } u \text{ dn } u$$

$$\frac{d\cos\phi}{du} = -\sin\phi\,\Delta\phi \qquad \text{or} \quad \frac{d\,\mathrm{cn}\,u}{du} = -\,\mathrm{sn}\,u\,\mathrm{dn}\,u$$

$$\frac{d\Delta\phi}{du} = -\,\kappa^2\sin\phi\cos\phi \quad \text{or} \quad \frac{d\,\mathrm{dn}\,u}{du} = -\,\kappa^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u$$

11.11. The complete integral over the quadrant, $0 < \phi < \frac{\pi}{2}$, $0 < x < 1$, defines the (quarter) period, K,

$$K = F\frac{\pi}{2} = \int_0^{\frac{1}{2}\pi}\frac{d\phi}{\Delta\phi},$$

making

$$\mathrm{sn}\;K = 1$$
$$\mathrm{cn}\;K = 0$$
$$\mathrm{dn}\;K = \kappa'.$$

κ' is the comodulus to κ, $\kappa^2 + \kappa'^2 = 1$, and the coperiod, K', is,

$$K' = \int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{(1-\kappa'^2\sin^2\phi)}}.$$

11.12.

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$$
$$\mathrm{cn}^2 u + \kappa^2\,\mathrm{sn}^2 u = 1$$
$$\mathrm{dn}^2 u - \kappa^2\,\mathrm{cn}^2 u = \kappa'^2.$$
$$\mathrm{sn}\;0 = 0, \quad \mathrm{cn}\;0 = \mathrm{dn}\;0 = 1.$$
$$\mathrm{sn}\,K = 1, \quad \mathrm{cn}\,K = 0, \qquad \mathrm{dn}\,K = \kappa'.$$

11.13. Legendre has calculated for every degree of θ, the modular angle, $\kappa = \sin\theta$, the value of $F\phi$ for every degree in the quadrant of the amplitude ϕ, and tabulated them in his Table IX, Fonctions elliptiques, t. II, $90 \times 90 = 8100$ entries.

But in this new arrangement of the Table, we take $u = F\phi$ as the independent variable of equal steps, and divide it into 90 degrees of a quadrant K, putting

$$u = cK = \frac{r^\circ}{90^\circ}K, \qquad r^\circ = 90^\circ c.$$

As in the ordinary trigonometrical tables, the degrees of r run down the left of the page from 0° to 45°, and rise up again on the right from 45° to 90°. Then columns II, III, X, XI are the equivalent of Legendre's Table of $F\phi$ and ϕ, but rearranged so that $F\phi$ proceeds by equal increments 1° in r°, and the increments in ϕ are unequal, whereas Legendre took equal increments of ϕ giving unequal increments in $u = F\phi$.

The reason of this rearrangement was the great advance made in elliptic function theory when Abel pointed out that $F\phi$ was of the nature of an inverse function, as it would be in a degenerate circular integral with zero modular angle. On Abel's recommendation, the notation is reversed, and ϕ is to be

considered a function of u, denoted already by $\phi = \text{am } u$, instead of looking at u, in Legendre's manner, as a function, $F\phi$, of ϕ. Jacobi adopted the idea in his Fundamenta nova, and employs the elliptic functions

$$\sin\phi = \sin \text{am } u, \qquad \cos\phi = \cos \text{am } u, \qquad \Delta\phi = \Delta \text{ am } u,$$

single-valued, uniform, periodic functions of the argument u, with (quarter) period K, as ϕ grows from 0 to $\frac{1}{2}\pi$. Gudermann abbreviated this notation to the one employed usually today.

11.2. The E. I. I is encountered in its simplest form, not as the elliptic arc, but in the expression of the time in the pendulum motion of finite oscillation, unrestricted to the small invisible motion of elementary treatment.

The compound pendulum, as of a clock, is replaced by its two equivalent particles, one at O in the centre of suspension, and the other at the centre of oscillation, P; the particles are adjusted so as to have the same total weight as the pendulum, the same centre of gravity at G, and the same moment of inertia about G or O; the two particles, if rigidly connected, are then the kinetic equivalent of the compound pendulum and move in the same way in the same field of force (Maxwell, Matter and Motion, CXXI).

Putting $OP = l$, called the simple equivalent pendulum length, and P starting from rest at B, in Figure 1, the particle P will move in the circular arc BAB' as if sliding down a smooth curve; and P will acquire the same velocity as if it fell vertically $KP = ND$; this is all the dynamical theory required.

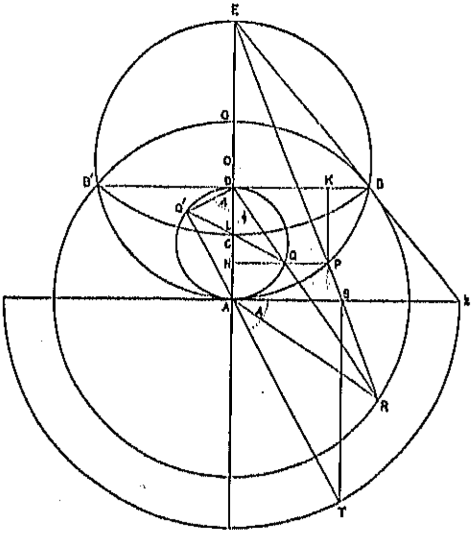

FIG. 1

$$(\text{velocity of } P)^2 = 2g \cdot KP,$$

$$(\text{velocity of } N)^2 = 2g \cdot ND \cdot \sin^2 AOP$$

$$= 2g \cdot ND \cdot \frac{NP^2}{OP^2} = \frac{g^2}{l^2} \cdot ND \cdot NA \cdot NE,$$

and with $AD = h$, $AN = y$, $ND = h - y$, $AE = 2l$, $NE = 2l - y$,

$$\left(\frac{dy}{dt}\right)^2 = \frac{2g}{l^2}(hy - y^2)(2l - y) = \frac{2g}{l^2} Y,$$

where Y is a cubic in y. Then t is given by an elliptic integral of the form $\int \frac{dy}{\sqrt{Y}}$. This integral is normalised to Legendre's standard form of his E. I. I by putting $y = h \sin^2\phi$, making $AOQ = \phi$, $h - y = h \cos^2\phi$, $2l - y = 2l(1 - \kappa^2 \sin^2\phi)$,

$$\kappa^2 = \frac{h}{2l} = \frac{AD}{AE} = \sin^2 AEB.$$

κ is called the modulus, AEB the modular angle which Legendre denoted by θ; $\sqrt{(1 - \kappa^2 \sin^2\phi)}$ he denoted by $\Delta\phi$.

With $g = ln^2$, and reckoning the time t from A, this makes

$$nt = \int_0^{\phi} \frac{d\phi}{\Delta\phi} = F\phi,$$

in Legendre's notation. Then the angle ϕ is called the amplitude of nt, to be denoted am nt, the particle P starting up from A at time $t = 0$; and with $u = nt$,

$$\operatorname{sn} u = \frac{AP}{AB} = \frac{AQ}{AD} \qquad \operatorname{sn}^2 u = \frac{AN}{AD}$$

$$\operatorname{cn} u = \frac{DQ}{AD} \qquad \operatorname{cn}^2 u = \frac{PK}{AD}$$

$$\operatorname{dn} u = \frac{EP}{EA} \qquad \operatorname{dn}^2 u = \frac{NE}{AE}$$

Velocity of $P = n \cdot AB \cdot \operatorname{cn} u = \sqrt{BP \cdot PB'}$, with an oscillation beat of T seconds in $u = eK$, $e = 2t/T$.

11.21. The numerical values of sn, cn, dn, tn (u, κ) are taken from a table to modulus $\kappa = \sin$ (modular angle, θ) by means of the functions Dr, Ar, Br, Cr, in columns V, VI, VII, VIII, by the quotients,

$$\sqrt{\kappa'} \operatorname{sn} eK = \frac{A}{D}$$

$$\operatorname{cn} eK = \frac{B}{D}$$

$$\frac{\operatorname{dn} eK}{\sqrt{\kappa'}} = \frac{C}{D}$$

$$\sqrt{\kappa'} \operatorname{tn} eK = \frac{A}{B}$$

$$r^\circ = 90^\circ e$$

$$u = eK.$$

These D, A, B, C are the Theta Functions of Jacobi, normalised, defined by

$$D(r) = \frac{\Theta u}{\Theta o}, \qquad A(r) = \frac{Hu}{HK},$$

$$B(r) = A(90^\circ - r) \qquad C(r) = D(90^\circ - r).$$

They were calculated from the Fourier series of angles proceeding by multiples of r°, and powers of q as coefficients, defined by

$$q = e^{-\pi \frac{K'}{K}}$$

$$\Theta u = 1 - 2q \cos 2r + 2q^4 \cos 4r - 2q^9 \cos 6r + \dots.$$

$$Hu = 2q^{\frac{1}{4}} \sin r - 2q^{\frac{9}{4}} \sin 3r + 2q^{\frac{25}{4}} \sin 5r - \dots.$$

11.3. The Elliptic Integral of the Second Kind (E. I. II) arose first historically in the rectification of the ellipse, hence the name. With $BOP = \phi$ in Figure 2, the minor eccentric angle of P, and s the arc BP from B to P at $x = a \sin \phi$, $y = b \cos \phi$,

$$\frac{ds}{d\phi} = \sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a\Delta(\phi, \kappa),$$

to the modulus κ, the eccentricity of the ellipse. Then $s = a\,E\phi$, where $\int_0^\phi \Delta\phi \cdot d\phi$ is denoted by $E\phi$ in Legendre's notation of his standard E. I. II; it is tabulated in his Table IX alongside of $F\phi$ for every degree of the modular angle θ, and to every degree in the quadrant of the amplitude ϕ.

But it is not possible to make the inversion and express ϕ as a single-valued function of $E\phi$.

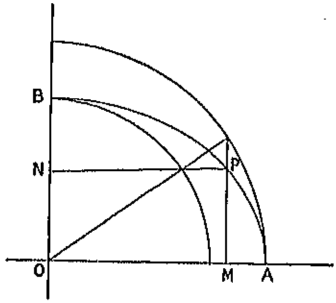

FIG. 2

11.31. The E. I. II, $E\phi$, arises also in the expression of the time, t, in the oscillation of a particle, P, on the arc of a parabola, as $F\phi$ was required on the arc of a circle. Starting from B along the parabola BAB', Figure 3, and with $AO = h$, $OB = b$, $BOQ = \phi$, $AN = y = h\cos^2\phi$, $NP = x = b\cos\phi$ and with $OS = 2h = b\tan\alpha$, $OA' = SB = b\sec\alpha$, the parabola cutting the horizontal at B at an angle α, the modular angle, $BRA'B'$ is a semi-ellipse, with focus at S, and eccentricity $\kappa = \sin\alpha$.

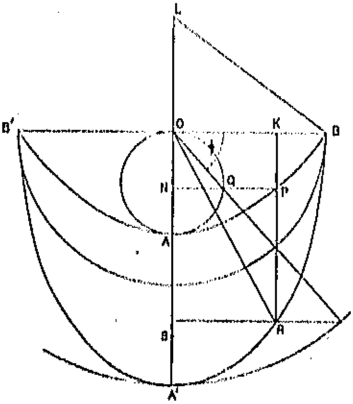

FIG. 3

$$(\text{Velocity of } P)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= (b^2\cos^2\phi + 4h^2\sin^2\phi\cos^2\phi)\left(\frac{d\phi}{dt}\right)^2$$

$$= a^2(1 - \sin^2\alpha\sin^2\phi)\cos^2\phi\left(\frac{d\phi}{dt}\right)^2 = 2gy = 2gh\cos^2\phi$$

$$= V^2\cos^2\phi,$$

if V denotes the velocity of P at A, and $OA' = a$. Then with s the elliptic arc BR,

$$V\frac{dt}{d\phi} = a\Delta\phi = a\frac{ds}{d\phi},\quad Vt = s,$$

and so the point R moves round the ellipse with constant velocity V, and accompanies the point P on the same vertical, oscillating on the parabola from B to B'.

In the analogous case of the circular pendulum, the time t would be given by the arc of an Elastica, in Kirchhoff's Kinetic Analogue, and this can be placed as a bow on Figure 1, with the cord along AE and vertex at B.

Legendre has shown also how in the oscillation of R on the semi-ellipse BRB' in a gravity field the time t is expressible by elliptic integrals, two of the first and two of the second kind, to complementary modulus (Fonctions elliptiques, I, p. 183).

11.32. In these tables, $E\phi$ is replaced by the columns IV, IX, of $E(r)$ and $G(r) = E(90 - r)$, defined, in Jacobi's notation, by

$$E(r) = \text{zn}\, cK = E\phi - cE$$
$$G(r) = \text{zn}\,(1 - c)K, \quad r = 90c.$$

This is the periodic part of $E\phi$ after the secular term $cE = \frac{E}{K}u$ has been set aside, E denoting the complete E. I. II,

$$E = E\tfrac{1}{2}\pi = \int^{\frac{1}{2}\pi}\Delta\phi\cdot d\phi.$$

The function zn u, or Zu in Jacobi's notation, or $E(r)$ in our notation, is calculated from the series,

$$Er = Zu = \frac{\pi}{K}\sum_{m=1}^{\infty}\frac{\sin 2mr}{\sinh m\pi\dfrac{K'}{K}} = \frac{2\pi}{K}\sum_{m=1}^{\infty}(q^m + q^{3m} + q^{5m} + \ldots)\sin 2mr.$$

This completes the explanation of the twelve columns of the tables.

11.4. The Double Periodicity of the Elliptic Functions.

This can be visualised in pendulum motion if gravity is supposed reversed suddenly at B (Figure 1) the end of a swing; as if by the addition of a weight to bring the centre of gravity above O, or by the movement of a weight, as in the metronome. The point P then oscillates on the arc BEB', and beats the elliptic function to the complementary modulus κ', as if in imaginary time, to imaginary argument $nti = fK'i$: and it reaches P' on AX produced, where tan AEP' = tan $AEB\cdot$cn $(nt'i, \kappa)$, or tan EAP' = tan $EAB\cdot$cn (nt', κ'); or with $nt' = v$, $DR' = DB\cdot$cn (iv, κ'), $DR = DB\cdot$cn (v, κ'), with $DR\cdot DR' = DB^2$, EP' crossing DB in R'.

$$\text{cn}\,(iv, \kappa) = \frac{1}{\text{cn}\,(v, \kappa')}$$

$$\text{sn}\,(iv, \kappa) = \frac{i\,\text{sn}\,(v, \kappa')}{\text{cn}\,(v, \kappa')} = i\,\text{tn}\,(v, \kappa')$$

$$\text{dn}\,(iv, \kappa) = \frac{\text{dn}\,(v, \kappa')}{\text{cn}\,(v, \kappa')} = \frac{1}{\text{sn}\,(K' - v, \kappa')}$$

where K' denotes the complementary (quarter) period to comodulus κ'.

If m, m' are any integers, positive or negative, including 0,

$$\begin{aligned}
&\text{sn}\,(u + 4mK + 2m'iK') &&= \text{sn}\,u\\
&\text{cn}\,[u + 4mK + 2m'(K + iK')] &&= \text{cn}\,u\\
&\text{dn}\,(u + 2mK + 4m'iK') &&= \text{dn}\,u
\end{aligned}$$

11.41. The Addition Theorem of the Elliptic Functions.

$$\text{sn}\,(u \pm v) = \frac{\text{sn}\,u\,\text{cn}\,v\,\text{dn}\,v \pm \text{sn}\,v\,\text{cn}\,u\,\text{dn}\,u}{1 - \kappa^2\,\text{sn}^2 u\,\text{sn}^2 v}$$

$$\text{cn}\,(v \pm u) = \frac{\text{cn}\,u\,\text{cn}\,v \mp \text{sn}\,u\,\text{dn}\,u\,\text{sn}\,v\,\text{dn}\,v}{1 - \kappa^2\,\text{sn}^2 u\,\text{sn}^2 v}$$

$$\text{dn}\,(v \pm u) = \frac{\text{dn}\,u\,\text{dn}\,v \mp \kappa^2\,\text{sn}\,u\,\text{cn}\,u\,\text{sn}\,v\,\text{cn}\,v}{1 - \kappa^2\,\text{sn}^2 u\,\text{sn}^2 v}$$

11.42. Coamplitude Formulas, with $v = \pm K$,

$$\operatorname{sn}(K-u) = \frac{\operatorname{cn} u}{\operatorname{dn} u} = \operatorname{sn}(K+u)$$

$$\operatorname{cn}(K-u) = \frac{\kappa' \operatorname{sn} u}{\operatorname{dn} u} \qquad \operatorname{cn}(K+u) = -\frac{\kappa' \operatorname{sn} u}{\operatorname{dn} u}$$

$$\operatorname{dn}(K-u) = \frac{\kappa'}{\operatorname{dn} u} = \operatorname{dn}(K+u)$$

$$\operatorname{tn}(K-u) = \frac{1}{\kappa' \operatorname{tn} u} \qquad \operatorname{tn}(K+u) = -\frac{1}{\kappa' \operatorname{tn} u}$$

11.43. Legendre's Addition Formula for his E. I. II,

$$E\phi = \int \Delta\phi \cdot d\phi = \int \operatorname{dn}^2 u \cdot du, \qquad \phi = \int \operatorname{dn} u \cdot du = \operatorname{am} u.$$

$$E\phi + E\psi - E\sigma = \kappa^2 \sin\phi \sin\psi \sin\sigma, \ \psi = \operatorname{am} v, \ \sigma = \operatorname{am}(v+u)$$

or, in Jacobi's notation,

$$\operatorname{zn} u + \operatorname{zn} v - \operatorname{zn}(u+v) = \kappa^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(v+u),$$

the secular part cancelling.

Another form of the Addition Theorem for Legendre's E. I. II,

$$E\sigma - E\theta - 2E\psi = \frac{-2\kappa^2 \sin\psi \cos\psi \, \Delta\psi \sin^2\phi}{1 - \kappa^2 \sin^2\phi \sin^2\psi}, \ \theta = \operatorname{am}(v-u)$$

or, in Jacobi's notation,

$$\operatorname{zn}(v+u) + \operatorname{zn}(v-u) - 2\operatorname{zn} v = \frac{-2\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}.$$

11.5. The Elliptic Integral of the Third Kind (E. I. III) is given by the next integration with respect to u, and introduces Jacobi's Theta Function, Θu, defined by,

$$\frac{d \log \Theta u}{du} = Zu = \operatorname{zn} u$$

$$\frac{\Theta u}{\Theta 0} = \exp. \int_0 \operatorname{zn} u \cdot du.$$

Integrating then with respect to u,

$$\log \Theta(v+u) - \log \Theta(v-u) - 2u \operatorname{zn} v = \int_0 \frac{-2\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} du,$$

and this integral is Jacobi's standard form of the E. I. III, and is denoted by $-2\Pi(u, v)$; thus,

$$\Pi(u, v) = \int \frac{\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} du = u \operatorname{zn} v + \tfrac{1}{2} \log \frac{\Theta(v-u)}{\Theta(v+u)}.$$

Jacobi's Eta Function, $\mathrm{H} v$, is defined by

$$\frac{\mathrm{H} v}{\Theta v} = \sqrt{\kappa} \operatorname{sn} v,$$

and then

$$\frac{d \log \mathrm{H} v}{dv} = \frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} + \operatorname{zn} v, \text{ denoted by } \operatorname{zs} v;$$

so that

$$\int_0 \frac{\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}\, du}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = u \frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v} + \Pi(u, v)$$

$$= u \operatorname{zs} v + \frac{1}{2} \log \frac{\Theta(v - u)}{\Theta(v + u)}$$

$$= \frac{1}{2} \log \frac{\Theta(v - u)}{\Theta(v + u)} e^{2u \operatorname{zs} v}$$

This gives Legendre's standard E. I. III,

$$\int^{\phi} \frac{M}{1 + n \sin^2 \phi} \frac{d\phi}{\Delta\phi},$$

where we put $n = -\kappa^2 \operatorname{sn}^2 v = -\kappa^2 \sin^2 \psi$,

$$M^2 = -\left(1 + \frac{\kappa^2}{n}\right)(1 + n) = \frac{\cos^2 \psi \Delta^2 \psi}{\sin^2 \psi} = \frac{\operatorname{cn}^2 v \operatorname{dn}^2 v}{\operatorname{sn}^2 v};$$

the normalising multiplier, M.

The E. I. III arises in the dynamics of the gyroscope, top, spherical pendulum, and in Poinsot's herpolhode. It can be visualized in the solid angle of a slant cone, or in the perimeter of the reciprocal cone, a sphero-conic, or in the magnetic potential of the circular base.

11.51. We arrive here at the definitions of the functions in the tables. Jacobi's Θu and $\mathrm{H}u$ are normalised by the divisors Θo and $\mathrm{H}K$, and with $r = 90e$,

$$D(r) \text{ denotes } \frac{\Theta eK}{\Theta K}, \qquad A(r) \text{ denotes } \frac{\mathrm{H}eK}{\mathrm{H}K}$$

while $B(r) = A(90 - r)$, $C(r) = D(90 - r)$, and $B(0) = A(90) = D(0) = C(90)$ $= 1$, $C(0) = D(90) = \frac{1}{\sqrt{\kappa}}$.

Then in the former definitions,

$$\frac{A(r)}{D(r)} = \frac{A(90)}{D(90)} \operatorname{sn} u = \sqrt{\kappa'} \operatorname{sn} eK$$

$$\frac{B(r)}{D(r)} = \frac{B(0)}{D(0)} \operatorname{cn} u = \operatorname{cn} eK$$

$$\frac{C(r)}{D(r)} = \frac{C(0)}{D(0)} \operatorname{dn} u = \frac{\operatorname{dn} eK}{\sqrt{\kappa'}}.$$

Then, with $u = eK$, $v = fK$, $r = 90e$, $s = 90f$,

$$(u, v) = eK \operatorname{zn} fK + \frac{1}{2} \log \frac{\Theta(f - e) K}{\Theta(f + e) K}$$

$$= eK\, E(s) + \frac{1}{2} \log \frac{D(s - r)}{D(s + r)}$$

$$\operatorname{zn} fK = E(s), \qquad \operatorname{zn}(1 - f) K = E(90 - s) = G(s).$$

The Jacobian multiplication relations of his theta functions can then be rewritten

$$D(r+s)D(r-s) = D^2rD^2s - \tan^2\theta A^2rA^2s,$$
$$A(r+s)A(r-s) = A^2rD^2s - D^2rA^2s,$$
$$B(r+s)B(r-s) = B^2rB^2s - A^2rA^2s.$$

But unfortunately for the physical applications the number s proves usually to be imaginary or complex, and Jacobi's expression is useless; Legendre calls this the circular form of the E. I. III, the logarithmic or hyperbolic form corresponding to real s. However, the complete E. I. III between the limits $0<\phi<\frac{1}{2}\pi$, or $0<u<K$, $0<c<1$, can always be expressed by the E. I. I and II, as Legendre pointed out.

11.6. The standard forms are given above to which an elliptic integral must be reduced when the result is required in a numerical form taken from the Tables. But in a practical problem the integral arises in a general algebraical form, and theory shows that the result can always be made, by a suitable substitution, to depend on three differential elements, of the I, II, III kind,

$$\text{I} \quad \frac{ds}{\sqrt{S}}$$

$$\text{II} \quad (s-a)\frac{ds}{\sqrt{S}}$$

$$\text{III} \quad \frac{1}{(s-\sigma)}\frac{ds}{\sqrt{S}}$$

where S is a cubic in the variable s which may be written, when resolved into three factors,

$$S = 4(s-s_1)(s-s_2)(s-s_3)$$

in the sequence $\infty > s_1 > s_2 > s_3 > -\infty$, and normalised to a standard form of zero degree these differential elements are

$$\text{I} \quad \frac{\sqrt{s_1-s_3}\,ds}{\sqrt{S}}$$

$$\text{II} \quad \frac{s-a}{\sqrt{s_1-s_3}}\frac{ds}{\sqrt{S}}$$

$$\text{III} \quad \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}}$$

Σ denoting the value of S when $s=\sigma$.

The relative positions of s and σ in the intervals of the sequence require preliminary consideration before introducing the Elliptic Functions and their notation.

11.7. For the E. I. I and its representation in a tabular form with

$$\kappa^2 = \frac{s_2 - s_3}{s_1 - s_3}, \qquad \kappa'^2 = \frac{s_1 - s_2}{s_1 - s_3},$$

$$K = \int_{s_1, s_3}^{\infty, s_2} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}}, \qquad K' = \int_{s_2, -\infty}^{s_1, s_3} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}},$$

and utilizing the inverse notation, then in the first interval of the sequence,

$$\infty > s > s_1$$

$$cK = \int_s^\infty \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s - s_1}{s - s_3}} = \mathrm{dn}^{-1}\sqrt{\frac{s - s_2}{s - s_3}}$$

$$(1 - c)K = \int_{s_1}^s \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_1}{s - s_2}} = \mathrm{cn}^{-1}\sqrt{\frac{s_1 - s_2}{s - s_2}} = \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s_2 \cdot s - s_3}{s_1 - s_3 \cdot s - s_2}}$$

indicating the substitutions,

$$\frac{s_1 - s_3}{s - s_3} = \sin^2\phi = \mathrm{sn}^2\, cK, \qquad \frac{s - s_1}{s - s_2} = \sin^2\psi = \mathrm{sn}^2\,(1 - c)K.$$

In the next interval S is negative, and the comodulus κ' is required.

$$s_1 > s > s_2$$

$$fK' = \int_s^{s_1} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_2}} = \mathrm{cn}^{-1}\sqrt{\frac{s - s_2}{s_1 - s_2}} = \mathrm{dn}^{-1}\sqrt{\frac{s - s_3}{s_1 - s_3}}.$$

$$(1 - f)K' = \int_{s_2}^s \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3 \cdot s - s_2}{s_1 - s_2 \cdot s - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s_3 \cdot s_1 - s}{s_1 - s_2 \cdot s - s_3}}$$

$$= \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s_3}{s - s_3}}$$

S is positive again in the next interval, and the modulus is κ.

$$s_2 > s > s_3$$

$$(1 - c)K = \int_s^{s_2} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3 \cdot s_2 - s}{s_2 - s_3 \cdot s_1 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_1 - s_2 \cdot s - s_3}{s_2 - s_3 \cdot s_1 - s}}$$

$$= \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s_2}{s_1 - s}}$$

$$cK = \int_{s_3}^s \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = \mathrm{sn}^{-1}\sqrt{\frac{s - s_3}{s_2 - s_3}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s}{s_2 - s_3}} = \mathrm{dn}^{-1}\sqrt{\frac{s_1 - s}{s_1 - s_3}}$$

indicating the substitutions,

$$\frac{s_1 - s_2}{s_1 - s} = \Delta^2\psi = \mathrm{dn}^2\,(1 - c)K, \qquad \frac{s - s_3}{s_2 - s_3} = \sin^2\phi = \mathrm{sn}^2\, cK$$

$$s = s_2 \sin^2\phi + s_3 \cos^2\phi$$

S is negative again in the last interval, and the modulus κ'.

$$s_3 > s > -\infty$$

$$(1-f)K' = \int_s^{s_3} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_3 - s}{s_2 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_2 - s_3}{s_2 - s}} = \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s_3 \cdot s_1 - s}{s_1 - s_3 \cdot s_2 - s}}$$

$$fK' = \int_{-\infty}^{s} \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{-S}} = \mathrm{sn}^{-1}\sqrt{\frac{s_1 - s_3}{s_1 - s}} = \mathrm{cn}^{-1}\sqrt{\frac{s_3 - s}{s_1 - s}} = \mathrm{dn}^{-1}\sqrt{\frac{s_2 - s}{s_1 - s}}$$

11.8. For the notation of the E. I. II and the various reductions, take the treatment given in the Trans. Am. Math. Soc., 1907, vol. 8, p. 450. The Jacobian Zeta Function and the Er, Gr of the Tables, are defined by the standard integral

$$\int_{s_3}^{s} \frac{s_1 - s}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \int_0^{\phi} \Delta\phi \cdot d\phi = E\phi = \int_0^{e} \mathrm{dn}^2(eK) \cdot d(eK) = E \,\mathrm{am}\, eK = eH + \mathrm{zn}\, eK,$$

or,

$$\int_{s_3}^{\sigma} \frac{\sigma - s_3}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int_0^{f} \mathrm{dn}^2 (fK') \cdot d(fK') = E \,\mathrm{am}\, fK' = fH' + \mathrm{zn}\, fK',$$

where zn is Jacobi's Zeta Function, and H, H' the complete E. I. II to modulus κ, κ', defined by,

$$H = \int_0^{\frac{\pi}{2}} \Delta(\phi, \kappa)\, d\phi = \int_0^1 \mathrm{dn}^2 (eK) \cdot d(eK)$$

$$H' = \int_0^{\frac{\pi}{2}} \Delta(\phi, \kappa')\, d\phi = \int_0^1 \mathrm{dn}^2 (fK') \cdot d(fK').$$

The function $\mathrm{zn}\, u$ is derived by logarithmic differentiation of Θu,

$\mathrm{zn}\, u = \dfrac{d \log \Theta u}{du}$, or concisely,

$$\Theta u = \exp. \int \mathrm{zn}\, u \cdot du,$$

and a function $\mathrm{zs}\, u$ is derived similarly from

$$\begin{aligned} \mathrm{zs}\, u &= \frac{d \log Hu}{du} \\ &= \frac{d \log \Theta u}{du} + \frac{d \log \mathrm{sn}\, u}{du} \\ &= \mathrm{zn}\, u + \frac{\mathrm{cn}\, u \,\mathrm{dn}\, u}{\mathrm{sn}\, u}. \end{aligned}$$

For the incomplete E. I. II in the regions,

$$\infty > s > s_1 > s_2 > s > s_3$$

and

$$\mathrm{sn}^2\, eK = \frac{s_1 - s_3}{s - s_3} \text{ or } \frac{s - s_3}{s_2 - s_3},$$

$$\int_s^{s_1} \frac{s - s_1}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \int_s^{s_2} \frac{s_2 - s}{s - s_3} \frac{\sqrt{s - s_3}}{\sqrt{S}}\, ds = -(1 - e)H + \text{zs}\, eK$$

$$\int \frac{s - s_2}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \kappa^2 \int \frac{s_1 - s}{s - s_3} \frac{\sqrt{s_1 - s_3}}{\sqrt{S}}\, ds = -(1 - e)(H - \kappa'^2 K) + \text{zs}\, eK$$

$$\int \frac{s - s_3}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{s_1 - s_3}}{\sqrt{S}}\, ds = (1 - e)(K - H) + \text{zs}\, eK$$

the integrals being ∞ at the upper limit, $s = \infty$, or at the lower limit, $s = s_3$ where $e = 0$ and $\text{zs}\, eK = \infty$.

So also,

$$\int_{s,\, s_1}^{\infty,\, s} \frac{s - s_2}{s - s_3} \frac{\sqrt{s_1 - s_3}}{\sqrt{S}}\, ds = \int_{s_3,\, s}^{s,\, s_2} \frac{s_1 - s}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \begin{array}{l} eH + \text{zn}\, eK \\ (1 - e)H - \text{zn}\, eK \end{array}$$

$$\int \frac{s - s_1}{s - s_3} \frac{\sqrt{s_1 - s_3}}{\sqrt{S}}\, ds = \int \frac{s_2 - s}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \begin{array}{l} e(H - \kappa'^2 K) + \text{zn}\, eK \\ (1 - e)(H - \kappa'^2 K) - \text{zn}\, eK \end{array}$$

$$\int \frac{s_2 - s_3}{s - s_3} \frac{\sqrt{s_1 - s_3}}{\sqrt{S}}\, ds = \int \frac{s - s_3}{\sqrt{s_1 - s_3}} \frac{ds}{\sqrt{S}} = \begin{array}{l} e(K - H) - \text{zn}\, eK \\ (1 - e)(K - H) + \text{zn}\, eK \end{array}$$

Similarly, for the variable σ in the regions

$$s_1 > \sigma > s_2 > s_3 > \sigma > -\infty$$

Σ negative, and

$$\text{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_3} \text{ or } \frac{s_1 - s_3}{s_1 - \sigma}$$

$$\int_{\sigma,\, s_2}^{s_1,\, \sigma} \frac{s_1 - \sigma}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int_{-\infty,\, \sigma}^{\sigma,\, s_3} \frac{s_1 - s_2}{s_1 - \sigma} \frac{\sqrt{s_1 - s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{array}{l} f(K' - H') - \text{zn}\, fK' \\ (1 - f)(K' - H') + \text{zn}\, fK' \end{array}$$

$$\int \frac{\sigma - s_2}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int \frac{s_3 - \sigma}{s_1 - \sigma} \frac{\sqrt{s_1 - s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{array}{l} f(H' - \kappa'^2 K') + \text{zn}\, fK' \\ (1 - f)(H' - \kappa'^2 K') - \text{zn}\, fK' \end{array}$$

$$\int \frac{\sigma - s_3}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = \int \frac{s_2 - \sigma}{s_1 - \sigma} \frac{\sqrt{s_1 - s_3}}{\sqrt{-\Sigma}}\, d\sigma = \begin{array}{l} fH' + \text{zn}\, fK' \\ (1 - f)H' - \text{zn}\, fK' \end{array}$$

$$\int_{s_1}^{\sigma} \frac{s_1 - s_2}{s_1 - \sigma} \frac{\sqrt{s_1 - s_3}}{\sqrt{-\Sigma}}\, d\sigma = \int_{\sigma}^{s_1} \frac{s_1 - \sigma}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = (1 - f)(K' - H') + \text{zs}\, fK'$$

$$\kappa'^2 \int \frac{s_3 - \sigma}{s_1 - \sigma} \frac{\sqrt{s_1 - s_3}}{\sqrt{-\Sigma}}\, d\sigma = \int \frac{s_2 - \sigma}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = -(1 - f)(H' - \kappa^2 K') + \text{zs}\, fK'$$

$$\int \frac{s_2 - \sigma}{s_1 - \sigma} \frac{\sqrt{s_1 - s_1}}{\sqrt{-\Sigma}}\, d\sigma = \int \frac{s_3 - \sigma}{\sqrt{s_1 - s_3}} \frac{d\sigma}{\sqrt{-\Sigma}} = -(1 - f)H' + \text{zs}\, fK'$$

these last three integrals being infinite at the upper limit, $\sigma = s_1$, or lower limit $\sigma = -\infty$, where $f = 0$, $\text{zs}\, fK' = \infty$.

Putting $e = 1$ or $f = 1$ any of these forms will give the complete E. I. II, noticing that zn K' and zs K' are zero.

11.9. In dealing practically with an E. I. III it is advisable to study it first in the algebraical form of Weierstrass,

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}\, ds}{(s-\sigma)\sqrt{S}},$$

where $S = 4 \cdot s - s_1 \cdot s - s_2 \cdot s - s_3$, Σ the same function of σ, and begin by examining the sequence of the quantities s, σ, s_1, s_2, s_3

Then in the region

$$s > s_1 > s_2 > \sigma > s_3,$$

put

$$s - s_3 = \frac{s_1 - s_3}{\operatorname{sn}^2 u},\ \sigma - s_3 = (s_2 - s_3)\operatorname{sn}^2 v,\ \kappa^2 = \frac{s_2 - s_3}{s_1 - s_3},$$

$$s - \sigma = \frac{s_1 - s_3}{\operatorname{sn}^2 u}(1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v),\ \frac{\sqrt{s_1 - s_3}\, ds}{\sqrt{S}} = du,$$

$$\sqrt{\Sigma} = \sqrt{s_1 - s_3}\,(s_2 - s_3)\operatorname{sn} v \operatorname{cn} v \operatorname{dn} v, \text{ making}$$

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = \int \frac{\kappa^2 \operatorname{sn} v \operatorname{cn} v \operatorname{dn} v \operatorname{sn}^2 u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}\, du = \Pi(u, v).$$

But in the region,

$$\sigma > s_1 > s_2 > s > s_3,$$

$$s - s_3 = (s_2 - s_3)\operatorname{sn}^2 u,\ \sigma - s_3 = \frac{s_1 - s_3}{\operatorname{sn}^2 v},\ \frac{1}{2}\sqrt{\Sigma} = (s_1 - s_3)^{\frac{3}{2}}\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn}^3 v},$$

$$\sigma - s = \frac{s_1 - s_3}{\operatorname{sn}^2 v}(1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v),$$

making,

$$\int \frac{\frac{1}{2}\sqrt{\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} = \int \frac{\dfrac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}\, du}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v} = \Pi_1 = \Pi(u, v) + u\frac{\operatorname{cn} v \operatorname{dn} v}{\operatorname{sn} v}.$$

In a dynamical application the sequence is usually

$$s > s_1 > \sigma > s_2 > s > s_3$$

or

$$s > s_1 > s_2 > s > s_3 > \sigma,$$

making Σ negative, and the E. I. III is then called circular; the parameter v is then imaginary, and the expression by the Theta function is illusory.

The complete E. I. III, however, was shown by Legendre to be tractable and falls into four classes, lettered (l') (m'), p. 138, (i'), (k'), pp. 133, 134 (Fonctions elliptiques, I).

$s_1 > \sigma > s_2$

$$\operatorname{sn}^2 fK' = \frac{s_1 - \sigma}{s_1 - s_2}$$

$$\operatorname{cn}^2 fK' = \frac{\sigma - s_2}{s_1 - s_2}$$

$$\operatorname{dn}^2 fK' = \frac{\sigma - s_3}{s_1 - s_3}$$

A. $$\infty > s > s_1, \quad \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = A(fK') = \tfrac{1}{2}\pi(1-f) - K\,\mathrm{zn}\,fK'$$

B. $$s_2 > s > s_3, \quad \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{\sigma - s}\frac{ds}{\sqrt{S}} = B(fK') = \tfrac{1}{2}\pi f + K\,\mathrm{zn}\,fK'$$

$$A + B = \tfrac{1}{2}\pi.$$

$s_3 > \sigma > -\infty$

$$\mathrm{sn}^2 fK' = \frac{s_1 - s_3}{s_1 - \sigma}$$

$$\mathrm{cn}^2 fK' = \frac{s_3 - \sigma}{s_1 - \sigma}$$

$$\mathrm{dn}^2 fK' = \frac{s_2 - \sigma}{s_1 - \sigma}$$

C. $$\infty > s > s_1, \quad \int_{s_1}^{\infty} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = C(fK') = K\,\mathrm{zn}\,fK' - \tfrac{1}{2}\pi(1-f)$$

D. $$s_2 > s > s_3, \quad \int_{s_3}^{s_2} \frac{\frac{1}{2}\sqrt{-\Sigma}}{s-\sigma}\frac{ds}{\sqrt{S}} = D(fK') = K\,\mathrm{zn}\,fK' + \tfrac{1}{2}\pi f$$

$$D - C = \tfrac{1}{2}\pi.$$

TABLES OF ELLIPTIC FUNCTIONS

By Col. R. L. Hippisley

K = 1.5737921309, K′ = 3.831742000, E = 1.5678090740, E′ = 1.012663506,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01748 65792	1	0	0.00006 64649	1.00000 05812	0.01745 23906
2	0.03497 31585	2	0	0.00013 28485	1.00000 23240	0.03489 94050
3	0.05245 97377	3	0	0.00019 90699	1.00000 52264	0.05233 59088
4	0.06994 63169	4	0	0.00026 50480	1.00000 92847	0.06975 64107
5	0.08743 28962	5	1	0.00033 07023	1.00001 44942	0.08715 56042
6	0.10491 94754	6	1	0.00039 59525	1.00002 08483	0.10452 84013
7	0.12240 60546	7	1	0.00046 07190	1.00002 83403	0.12186 92343
8	0.13989 26338	8	1	0.00052 49226	1.00003 69582	0.13917 29770
9	0.15737 92131	9	1	0.00058 84849	1.00004 66945	0.15643 43264
10	0.17486 57923	10	1	0.00065 13283	1.00005 75462	0.17364 80247
11	0.19235 23716	11	1	0.00071 33760	1.00006 94702	0.19080 88283
12	0.20983 89508	12	1	0.00077 45523	1.00008 24810	0.20791 15101
13	0.22732 55300	13	1	0.00083 47824	1.00009 65565	0.22495 08803
14	0.24481 21092	14	2	0.00089 39920	1.00011 16738	0.24192 16887
15	0.26229 86885	15	2	0.00095 21114	1.00012 78184	0.25881 88257
16	0.27978 52677	16	2	0.00100 90670	1.00014 49606	0.27563 71211
17	0.29727 18469	17	2	0.00106 47903	1.00016 31066	0.29237 14618
18	0.31475 84262	18	2	0.00111 92132	1.00018 22072	0.30901 67591
19	0.33224 50054	19	2	0.00117 22691	1.00020 22482	0.32556 78600
20	0.34973 15846	20	2	0.00122 38911	1.00022 32051	0.34201 98090
21	0.36721 81639	21	2	0.00127 40244	1.00024 50528	0.35836 76658
22	0.38470 47431	22	2	0.00132 25993	1.00026 77636	0.37460 63009
23	0.40219 13223	23	2	0.00136 95504	1.00029 13109	0.39073 08277
24	0.41967 79016	24	2	0.00141 48170	1.00031 56657	0.40673 63347
25	0.43716 44808	25	3	0.00145 84087	1.00034 07982	0.42261 79404
26	0.45465 10600	26	3	0.00150 01897	1.00036 66770	0.43837 08251
27	0.47213 76393	27	3	0.00154 01308	1.00039 32731	0.45399 01723
28	0.48962 42185	28	3	0.00157 82103	1.00042 05516	0.46947 12301
29	0.50711 07977	29	3	0.00161 43549	1.00044 84804	0.48480 92843
30	0.52459 73770	30	3	0.00164 85207	1.00047 70246	0.49999 96504
31	0.54208 39562	31	3	0.00168 06631	1.00050 61802	0.51503 77311
32	0.55957 05354	32	3	0.00171 08062	1.00053 58215	0.52991 89160
33	0.57705 71147	33	3	0.00173 88322	1.00056 60024	0.54463 89870
34	0.59454 36939	34	3	0.00176 47373	1.00059 66564	0.55919 25843
35	0.61203 02731	35	3	0.00178 84901	1.00062 77451	0.57357 60567
36	0.62951 68524	36	3	0.00181 00817	1.00065 92418	0.58778 49028
37	0.64700 34316	37	3	0.00182 94201	1.00069 10776	0.60181 46744
38	0.66449 00108	38	3	0.00184 65800	1.00072 32438	0.61566 11280
39	0.68197 65900	39	3	0.00186 14423	1.00075 56012	0.62931 90158
40	0.69946 31693	40	3	0.00187 40556	1.00078 83804	0.64278 73670
41	0.71694 97485	41	4	0.00188 43845	1.00082 12712	0.65605 85898
42	0.73443 63278	42	4	0.00189 24166	1.00085 44239	0.66913 03706
43	0.75192 29070	43	4	0.00189 81423	1.00088 74981	0.68199 80287
44	0.76940 94862	44	4	0.00190 15552	1.00092 07833	0.69465 80439
45	78689 60655	45	4	0.00190 26810	1.00095 40492	0.70710 64600
r	Fψ	ψ		G(r)	C(r)	B(r)

$q = 0.000476569016867$, $\Theta 0 = 0.9990468602$, $H(K) = 0.2955029021$

B(r)	C(r)	G(r)	ψ		Fψ	90°-r
1.00000 00000	1.00190 80084	0.00000 00000	90°	0′	1.57379 21309	90
0.99984 76940	1.00190 75172	0.00006 63384	89	0	1.55630 55517	89
0.99939 08250	1.00190 57743	0.00013 25961	88	0	1.53881 89724	88
0.99862 95323	1.00190 28720	0.00019 86928	87	0	1.52133 23932	87
0.99756 40458	1.00189 88136	0.00026 45481	86	0	1.50384 58140	86
0.99619 46912	1.00189 36042	0.00033 00820	85	1	1.48635 92347	85
0.99452 18855	1.00188 72501	0.00039 52149	84	1	1.46887 26555	84
0.99254 61382	1.00187 97590	0.00045 98676	83	1	1.45138 60763	83
0.99026 80513	1.00187 11401	0.00052 39616	82	1	1.43389 94971	82
0.98768 83186	1.00186 14039	0.00058 74190	81	1	1.41641 29178	81
0.98480 77200	1.00185 05621	0.00065 01626	80	1	1.39892 63386	80
0.98162 71810	1.00183 86282	0.00071 21163	79	1	1.38143 97593	79
0.97814 75643	1.00182 56165	0.00077 32046	78	1	1.36395 31801	78
0.97437 00200	1.00181 15429	0.00083 33534	77	1	1.34646 66009	77
0.97029 56747	1.00179 64246	0.00089 24894	76	2	1.32898 00217	76
0.96592 57675	1.00178 02800	0.00095 05409	75	2	1.31149 34424	75
0.96126 16906	1.00176 31288	0.00100 74371	74	2	1.29400 68632	74
0.95630 46817	1.00174 49018	0.00106 31089	73	2	1.27652 02840	73
0.95105 64338	1.00172 58912	0.00111 74885	72	2	1.25903 37047	72
0.94551 84840	1.00170 58502	0.00117 05097	71	2	1.24154 71255	71
0.93969 25200	1.00168 48032	0.00122 21081	70	2	1.22406 05463	70
0.93358 03176	1.00166 30459	0.00127 22208	69	2	1.20657 39670	69
0.92718 37304	1.00164 03347	0.00132 07868	68	2	1.18908 73878	68
0.92050 47258	1.00161 67874	0.00136 77470	67	2	1.17160 08086	67
0.91354 53203	1.00159 24327	0.00141 30440	66	3	1.15411 42293	66
0.90630 76400	1.00156 73003	0.00145 66228	65	3	1.13662 76501	65
0.89879 38804	1.00154 14205	0.00149 84301	64	3	1.11914 10709	64
0.89100 64574	1.00151 48252	0.00153 84151	63	3	1.10165 44916	63
0.88294 74161	1.00148 75407	0.00157 65289	62	3	1.08416 79124	62
0.87461 95204	1.00145 96182	0.00161 27250	61	3	1.06668 13332	61
0.86602 52071	1.00143 10738	0.00164 69592	60	3	1.04919 47539	60
0.85716 70941	1.00140 10481	0.00167 91897	59	3	1.03170 81747	59
0.84804 78708	1.00137 42768	0.00170 93771	58	3	1.01422 15955	58
0.83867 03419	1.00134 20059	0.00173 74846	57	3	0.99673 50162	57
0.82903 73370	1.00131 14423	0.00176 34776	56	3	0.97924 84370	56
0.81915 17995	1.00128 03532	0.00178 73244	55	3	0.96176 18578	55
0.80901 67404	1.00124 88666	0.00180 89958	54	3	0.94427 52785	54
0.79863 52473	1.00121 70208	0.00182 84651	53	3	0.92678 86993	53
0.78801 04823	1.00118 48546	0.00184 57085	52	3	0.90930 21201	52
0.77714 56818	1.00115 24072	0.00186 07047	51	3	0.89181 55409	51
0.76604 41556	1.00111 97181	0.00187 34353	50	3	0.87432 89616	50
0.75470 92851	1.00108 68272	0.00188 38846	49	3	0.85684 23824	49
0.74314 45232	1.00105 37745	0.00189 20395	48	3	0.83935 58031	48
0.73135 33926	1.00102 06003	0.00189 78900	47	3	0.82186 92239	47
0.71933 94850	1.00098 73450	0.00190 14287	46	4	0.80438 26447	46
0.70710 64600	1.00095 40492	0.00190 26510	45	4	0.78689 60655	45
A(r)	D(r)	E(r)	φ		Fφ	r

K = 1.5828428043, K′ = 3.153385252, E = 1.5588871966, E′ = 1.040114396,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01758 71423	1	0	0.00026 61187	1.00000 23404	0.01745 24509
2	0.03517 42845	2	1	0.00053 10998	1.00000 93587	0.03489 59861
3	0.05276 14268	3	1	0.00079 70418	1.00002 10464	0.05233 51018
4	0.07034 85691	4	2	0.00106 11070	1.00003 73900	0.06975 84570
5	0.08793 57113	5	2	0.00132 40433	1.00005 83670	0.08715 44758
6	0.10552 28536	6	3	0.00158 52573	1.00008 39546	0.10452 60489
7	0.12310 99959	7	3	0.00184 48182	1.00011 41206	0.12186 75840
8	0.14069 71382	8	4	0.00210 15060	1.00014 88284	0.13917 11019
9	0.15828 42804	9	4	0.00235 50064	1.00018 80350	0.15643 24298
10	0.17587 14227	10	5	0.00260 73014	1.00023 16915	0.17364 87109
11	0.19345 85650	11	5	0.00285 56013	1.00027 97510	0.19080 69043
12	0.21104 57072	12	5	0.00310 09619	1.00033 21401	0.20790 89771
13	0.22863 28495	13	6	0.00334 14153	1.00038 88224	0.22494 79201
14	0.24621 99918	14	6	0.00357 82585	1.00044 97058	0.24191 88505
15	0.26380 71340	15	7	0.00381 06020	1.00051 47160	0.25881 88080
16	0.28139 42763	16	7	0.00403 84304	1.00058 47959	0.27563 30452
17	0.29898 14186	17	7	0.00426 12186	1.00065 68104	0.29236 77084
18	0.31656 85609	18	8	0.00447 87507	1.00073 37362	0.30901 20004
19	0.33415 57031	19	8	0.00469 07873	1.00081 44094	0.32556 35012
20	0.35174 28454	20	8	0.00489 70511	1.00089 88432	0.34201 57107
21	0.36932 99877	21	9	0.00509 72061	1.00098 66106	0.35836 33745
22	0.38691 71299	22	9	0.00529 13778	1.00107 82064	0.37460 15764
23	0.40450 42722	23	9	0.00547 87506	1.00117 30898	0.39072 62791
24	0.42209 14145	24	10	0.00565 95131	1.00127 11647	0.40673 10711
25	0.43967 85568	25	10	0.00583 33188	1.00137 23717	0.42260 31771
26	0.45726 56990	26	10	0.00599 99643	1.00147 65874	0.43836 59507
27	0.47485 28413	27	11	0.00615 92488	1.00158 36015	0.45398 83206
28	0.49243 99836	28	11	0.00631 09780	1.00169 35336	0.46946 62009
29	0.51002 71258	29	11	0.00645 49624	1.00180 58998	0.48480 41881
30	0.52761 42681	30	11	0.00659 10184	1.00192 09461	0.49999 45073
31	0.54520 14104	31	12	0.00671 90513	1.00204 82111	0.51504 25321
32	0.56278 85526	32	12	0.00683 88242	1.00215 77178	0.52991 40530
33	0.58037 56949	33	12	0.00695 02342	1.00227 92542	0.54463 34430
34	0.59796 28372	34	12	0.00705 31150	1.00240 26911	0.55918 72240
35	0.61554 99795	35	12	0.00714 73709	1.00252 78880	0.57357 07999
36	0.63313 71217	36	13	0.00723 28968	1.00265 41826	0.58777 90173
37	0.65072 42640	37	13	0.00730 95745	1.00278 29236	0.60180 91908
38	0.66831 14063	38	13	0.00737 74166	1.00291 24548	0.61565 85750
39	0.68589 85485	39	13	0.00743 60969	1.00304 31184	0.62931 48240
40	0.70348 56908	40	13	0.00748 56962	1.00317 47551	0.64278 20847
41	0.72107 28331	41	13	0.00752 62073	1.00330 72046	0.65606 35558
42	0.73865 99754	42	13	0.00755 75345	1.00344 00986	0.66912 81036
43	0.75624 71176	43	13	0.00757 96433	1.00357 38980	0.68199 30169
44	0.77383 42599	44	13	0.00759 28102	1.00370 78127	0.69465 31058
45	0.79142 14022	45	13	0.00759 61238	1.00384 18628	0.70710 11626

K = 1.5981420021, K′ = K√3 = 2.7680631454, E = 1.6141501930, E′ = 1.076405113,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01775 71334	1	1	0.00059 97806	1.00000 53258	0.01745 10959
2	0.03551 42667	2	2	0.00119 88113	1.00002 12966	0.03489 68785
3	0.05327 14001	3	3	0.00179 63433	1.00004 78920	0.05233 20359
4	0.07102 85334	4	4	0.00239 16296	1.00008 50845	0.06975 12596
5	0.08878 56668	5	5	0.00298 30265	1.00013 28199	0.08714 93469
6	0.10654 28002	6	6	0.00357 24940	1.00019 10470	0.10452 06970
7	0.12429 99335	7	7	0.00415 65075	1.00025 96920	0.12186 03454
8	0.14205 70669	8	8	0.00473 55081	1.00033 86738	0.13916 28498
9	0.15981 42002	9	9	0.00530 85030	1.00042 78037	0.15642 30024
10	0.17757 13336	10	10	0.00587 48710	1.00052 72438	0.17363 55278
11	0.19532 84669	11	11	0.00643 39944	1.00063 66031	0.19079 51850
12	0.21308 56003	12	12	0.00698 49088	1.00075 58483	0.20789 67491
13	0.23084 27336	13	13	0.00752 71998	1.00088 48041	0.22493 50127
14	0.24859 98670	14	14	0.00806 01044	1.00102 33434	0.24190 47877
15	0.26635 70004	15	15	0.00858 29622	1.00117 12875	0.25880 09008
16	0.28411 41337	16	16	0.00909 51263	1.00132 84561	0.27561 82249
17	0.30187 12671	17	17	0.00959 50638	1.00149 46577	0.29235 16211
18	0.31962 84004	18	18	0.01008 48569	1.00166 96898	0.30899 59997
19	0.33738 55338	19	18	0.01056 12037	1.00185 33392	0.32554 69922
20	0.35514 26672	20	19	0.01102 44188	1.00204 53820	0.34199 74884
21	0.37289 98005	21	20	0.01147 39330	1.00224 55845	0.35834 44886
22	0.39065 69339	22	21	0.01190 91990	1.00245 37025	0.37458 24043
23	0.40841 40672	23	21	0.01232 96827	1.00266 94826	0.39070 62603
24	0.42617 12006	24	22	0.01273 48729	1.00289 26619	0.40671 11462
25	0.44392 83339	25	23	0.01312 42775	1.00312 29684	0.42259 21874
26	0.46168 54673	26	24	0.01349 74251	1.00336 01217	0.43834 48471
27	0.47944 26006	27	25	0.01385 38651	1.00360 38326	0.45396 34276
28	0.49719 97340	28	25	0.01419 31688	1.00385 38044	0.46944 40717
29	0.51495 68674	29	25	0.01451 49297	1.00410 97334	0.48478 17640
30	0.53271 40007	30	26	0.01481 87635	1.00437 13049	0.49997 18327
31	0.55047 11341	31	26	0.01510 43095	1.00463 82041	0.51500 96510
32	0.56822 82674	32	27	0.01537 12298	1.00491 01010	0.52989 06380
33	0.58598 54008	33	27	0.01561 92109	1.00518 66701	0.54461 02607
34	0.60374 25341	34	28	0.01584 79628	1.00546 75706	0.55916 40350
35	0.62149 96675	35	28	0.01605 72204	1.00575 24612	0.57354 75273
36	0.63925 68009	36	28	0.01624 67429	1.00604 09949	0.58775 63586
37	0.65701 39342	37	29	0.01641 63146	1.00633 28201	0.60178 61014
38	0.67477 10676	38	29	0.01656 57446	1.00662 75813	0.61563 27896
39	0.69252 82009	39	29	0.01669 48676	1.00692 49193	0.62929 18421
40	0.71028 53343	40	29	0.01680 35433	1.00722 44718	0.64275 92769
41	0.72804 24676	41	30	0.01689 16569	1.00752 58740	0.65603 09697
42	0.74579 96010	42	30	0.01695 91191	1.00782 87587	0.66910 28494
43	0.76355 67344	43	30	0.01700 58662	1.00813 27567	0.68197 09689
44	0.78131 38677	44	30	0.01703 18597	1.00843 74977	0.69463 13711
45	0.79907 10011	45	30	0.01703 70869	1.00874 26104	0.70708 92248

TABLE $\theta = 15°$

$q = 0.004333420500083$, $\Theta 0 = 0.0913331607$, $HK = 0.5131518035$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	1.01748 52237	0.00000 00000	90° 0′	1.59814 20021	90
0.99984 76723	1.01747 98079	0.00058 94801	89 1	1.58038 48688	89
0.99939 07350	1.01746 39271	0.00117 82606	88 2	1.56262 77354	88
0.99862 93293	1.01743 73307	0.00176 56424	87 3	1.54487 06021	87
0.99756 36857	1.01740 01412	0.00235 09281	86 4	1.52711 34687	86
0.99619 41297	1.01735 24037	0.00293 34228	85 5	1.50935 63353	85
0.99452 10702	1.01729 41766	0.00351 24342	84 6	1.49159 92020	84
0.99254 59444	1.01722 55307	0.00408 72741	83 7	1.47384 20686	83
0.99026 66280	1.01714 65406	0.00465 72589	82 8	1.45608 49353	82
0.98768 65251	1.01705 73207	0.00522 17102	81 9	1.43832 78019	81
0.98480 55245	1.01695 79708	0.00577 99557	80 10	1.42057 06685	80
0.98162 44000	1.01684 86202	0.00633 13300	79 11	1.40281 35352	79
0.97814 44248	1.01672 93840	0.00687 51750	78 12	1.38505 64019	78
0.97436 63613	1.01660 04100	0.00741 08412	77 13	1.36729 92685	77
0.97029 14608	1.01646 18796	0.00793 76880	76 14	1.34954 21352	76
0.96592 99661	1.01631 39354	0.00845 50845	75 15	1.33178 50018	75
0.96128 62102	1.01615 67668	0.00896 24102	74 16	1.31402 78684	74
0.95629 86138	1.01599 05651	0.00945 90560	73 17	1.29627 07351	73
0.95104 96047	1.01581 55420	0.00994 44245	72 18	1.27851 36017	72
0.94551 10478	1.01563 18834	0.01041 79308	71 18	1.26075 64684	71
0.93968 43642	1.01543 98405	0.01087 90033	70 19	1.24299 93350	70
0.93357 14207	1.01523 96380	0.01132 70844	69 20	1.22524 22016	69
0.92717 40818	1.01503 15108	0.01176 16310	68 20	1.20748 50683	68
0.92049 42975	1.01481 57306	0.01218 21151	67 21	1.18972 79349	67
0.91353 41057	1.01459 25602	0.01258 80246	66 22	1.17197 08016	66
0.90629 86284	1.01436 22536	0.01297 88640	65 23	1.15421 36682	65
0.89878 10728	1.01412 51003	0.01335 41547	64 23	1.13645 65348	64
0.89099 27403	1.01388 13802	0.01371 34350	63 24	1.11869 94015	63
0.88293 29750	1.01363 14174	0.01405 62649	62 25	1.10094 22681	62
0.87460 42661	1.01337 54893	0.01438 22180	61 25	1.08318 51348	61
0.86600 91414	1.01311 39167	0.01469 08906	60 26	1.06542 80014	60
0.85715 03210	1.01284 70184	0.01498 18982	59 26	1.04767 08681	59
0.84803 02088	1.01257 51105	0.01525 48767	58 27	1.02991 37347	58
0.83865 18817	1.01229 88512	0.01550 94825	57 27	1.01215 66014	57
0.82903 81005	1.01201 70807	0.01574 53930	56 28	0.99439 94680	56
0.81913 18020	1.01173 27599	0.01596 23105	55 28	0.97664 23346	55
0.80899 89997	1.01144 42262	0.01615 99545	54 28	0.95888 52013	54
0.79861 37846	1.01115 24009	0.01633 80704	53 29	0.94112 80679	53
0.78798 84184	1.01085 70397	0.01649 64258	52 29	0.92337 09346	52
0.77712 28430	1.01056 03017	0.01663 48119	51 29	0.90561 38012	51
0.76602 06601	1.01026 07491	0.01675 30432	50 29	0.88785 66678	50
0.75468 51808	1.00995 93468	0.01685 09584	49 29	0.87009 95345	49
0.74311 98430	1.00965 64622	0.01692 84205	48 30	0.85234 24011	48
0.73132 81506	1.00935 24642	0.01698 53170	47 30	0.83458 52678	47
0.71931 37274	1.00904 77232	0.01702 15600	46 30	0.81682 81344	46
0.70708 02248	1.00874 26104	0.01703 70869	45 30	0.79907 10011	45
A(r)	D(r)	E(r)	φ	Fφ	r

$K = 1.6200258991, \quad K' = 2.5045500790, \quad E = 1.5237002053, \quad E' = 1.118377738$

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01800 02878	1	2	0.00106 89581	1.00000 96218	0.01744 81883
2	0.03600 05755	2	4	0.00213 65522	1.00003 84757	0.03489 16604
3	0.05400 08633	3	6	0.00320 14202	1.00008 65263	0.05232 33377
4	0.07200 11511	4	7	0.00426 22042	1.00015 37152	0.06973 96909
5	0.09000 14388	5	9	0.00531 75510	1.00023 99608	0.08713 48413
6	0.10800 17266	6	11	0.00636 61180	1.00034 51572	0.10450 34078
7	0.12600 20144	7	13	0.00740 65708	1.00046 91770	0.12184 03160
8	0.14400 23021	8	15	0.00843 75848	1.00061 18680	0.13914 01051
9	0.16200 25899	9	17	0.00945 78515	1.00077 30591	0.15639 78097
10	0.18000 28777	10	19	0.01046 60772	1.00095 25810	0.17360 74610
11	0.19800 31655	11	20	0.01146 09855	1.00115 01262	0.19076 45434
12	0.21600 34532	12	22	0.01244 13188	1.00136 55138	0.20786 35073
13	0.23400 37410	13	24	0.01340 58406	1.00159 85414	0.22489 91208
14	0.25200 40288	14	25	0.01435 33370	1.00184 88451	0.24186 65298
15	0.27000 43165	15	27	0.01528 26180	1.00211 61266	0.25876 06626
16	0.28800 46043	16	28	0.01619 25107	1.00240 08704	0.27557 57786
17	0.30600 48921	17	30	0.01708 19057	1.00270 03408	0.29230 70609
18	0.32400 51799	18	32	0.01794 96684	1.00301 65642	0.30894 94182
19	0.34200 54676	19	33	0.01879 47304	1.00334 83565	0.32549 77555
20	0.36000 57554	20	35	0.01961 60466	1.00369 53431	0.34194 71266
21	0.37800 60431	21	36	0.02041 26046	1.00405 70112	0.35829 24349
22	0.39600 63309	22	37	0.02118 34268	1.00443 30010	0.37452 87349
23	0.41400 66187	23	39	0.02192 75711	1.00482 28518	0.39065 10841
24	0.43200 69064	24	40	0.02264 41331	1.00522 60614	0.40665 48753
25	0.45000 71942	25	41	0.02333 22426	1.00564 21475	0.42253 43354
26	0.46800 74820	26	42	0.02399 10740	1.00607 06033	0.43828 55296
27	0.48600 77697	27	44	0.02461 98478	1.00651 06067	0.45390 33615
28	0.50400 80575	28	45	0.02521 77802	1.00696 25213	0.46938 30764
29	0.52200 83453	29	46	0.02578 42130	1.00742 49698	0.48471 99582
30	0.54000 86330	30	46	0.02631 84541	1.00789 74700	0.49990 93470
31	0.55800 89208	31	47	0.02681 98888	1.00837 96651	0.51494 65858
32	0.57600 92086	32	48	0.02728 79396	1.00887 08936	0.52982 71240
33	0.59400 94963	33	49	0.02772 20732	1.00937 05600	0.54454 64181
34	0.61200 97841	34	50	0.02812 18009	1.00987 80525	0.55909 99845
35	0.63001 00719	35	50	0.02848 66791	1.01039 27530	0.57348 33858
36	0.64801 03597	36	51	0.02881 63091	1.01091 40471	0.58769 22416
37	0.66601 06474	37	51	0.02911 03382	1.01144 12069	0.60172 22208
38	0.68401 09352	38	52	0.02936 84591	1.01197 38011	0.61556 90470
39	0.70201 12230	39	52	0.02959 04103	1.01251 09908	0.62922 84964
40	0.72001 15107	40	53	0.02977 59763	1.01305 21815	0.64269 64140
41	0.73801 17985	41	53	0.02992 49873	1.01359 67148	0.65596 86845
42	0.75601 20863	42	53	0.03003 73198	1.01414 30245	0.66904 12642
43	0.77401 23740	43	53	0.03011 28953	1.01469 31466	0.68191 01665
44	0.79201 26618	44	53	0.03015 16811	1.01524 37112	0.69457 14668
45	0.81001 29496	45	53	0.03015 36896	1.01579 49474	0.70702 13033
90–r	Fψ	ψ		G(r)	C(r)	B(r)

$q = 0.0077740080416442$, $\Theta 0 = 0.9844506465$, $HK = 0.6939186400$

B(r)	C(r)	G(r)	ψ	Fψ	90–r
1.00000 00000	1.03158 99246	0.00000 00000	90° 0′	1.62002 58991	90
0.99984 76215	1.03158 03027	0.00103 62474	89 2	1.60202 56113	89
0.99939 05327	1.03155 14488	0.00207 12902	88 4	1.58402 53236	88
0.99862 88734	1.03150 33980	0.00310 39250	87 6	1.56602 50358	87
0.99756 28767	1.03143 62088	0.00413 29509	86 7	1.54802 47480	86
0.99619 28086	1.03134 99632	0.00515 71704	85 9	1.53002 44603	85
0.99451 92082	1.03124 47661	0.00617 53910	84 11	1.51202 41725	84
0.99254 25870	1.03112 07458	0.00718 64259	83 13	1.49402 38847	83
0.99026 34315	1.03097 80534	0.00818 90957	82 15	1.47602 35970	82
0.98768 24070	1.03081 68627	0.00918 22293	81 16	1.45802 33092	81
0.98480 05736	1.03063 73701	0.01016 46651	80 18	1.44002 30214	80
0.98161 85429	1.03043 97942	0.01113 52523	79 20	1.42202 27337	79
0.97813 73781	1.03022 43759	0.01209 28519	78 22	1.40402 24459	78
0.97435 81442	1.02999 13775	0.01303 63381	77 23	1.38602 21581	77
0.97028 10908	1.02974 10829	0.01396 45994	76 25	1.36802 18704	76
0.96591 01827	1.02947 37072	0.01487 65396	75 27	1.35002 15826	75
0.96124 40300	1.02918 98458	0.01577 10703	74 28	1.33202 12948	74
0.95628 40924	1.02888 95748	0.01664 71568	73 30	1.31402 10070	73
0.95103 45595	1.02857 33501	0.01750 37292	72 31	1.29602 07193	72
0.94549 43450	1.02824 15568	0.01833 97739	71 33	1.27802 04315	71
0.93969 60149	1.02789 45992	0.01915 42895	70 34	1.26002 01437	70
0.93358 44391	1.02753 28994	0.01994 62967	69 36	1.24201 98560	69
0.92718 23077	1.02715 69991	0.02071 48399	68 37	1.22401 95682	68
0.92047 08768	1.02676 70574	0.02145 89881	67 38	1.20601 92804	67
0.91350 80187	1.02636 38468	0.02217 78360	66 40	1.18801 89927	66
0.90626 86518	1.02594 77590	0.02287 05049	65 41	1.17001 87049	65
0.89875 22880	1.02551 93039	0.02353 61442	64 42	1.15201 84171	64
0.89099 21252	1.02507 89985	0.02417 39320	63 43	1.13401 81294	63
0.88290 05140	1.02462 73829	0.02478 30767	62 44	1.11601 78416	62
0.87457 00067	1.02416 50064	0.02536 28172	61 45	1.09801 75538	61
0.86597 30508	1.02369 24323	0.02591 24248	60 46	1.08001 72661	60
0.85711 23488	1.02321 02463	0.02643 12037	59 47	1.06201 69783	59
0.84799 08205	1.02271 90969	0.02691 84920	58 48	1.04401 66905	58
0.83861 04218	1.02221 93308	0.02737 36026	57 49	1.02601 64028	57
0.82897 48674	1.02171 18465	0.02779 61243	56 49	1.00801 61150	56
0.81908 68896	1.02119 71441	0.02818 53227	55 50	0.99001 58272	55
0.80894 94182	1.02067 58606	0.02854 07409	54 51	0.97201 55395	54
0.79856 88784	1.02014 86302	0.02886 19001	53 51	0.95401 52517	53
0.78793 88407	1.01961 69955	0.02914 83611	52 52	0.93601 49639	52
0.77707 18491	1.01907 89054	0.02939 97245	51 52	0.91801 46761	51
0.76596 70209	1.01853 77143	0.02961 56313	50 53	0.90001 43884	50
0.75463 10480	1.01799 31816	0.02979 57642	49 53	0.88201 41006	49
0.74306 43814	1.01744 59707	0.02993 98477	48 53	0.86401 38129	48
0.73127 14598	1.01689 67484	0.03004 76489	47 53	0.84601 35251	47
0.71925 58784	1.01634 61837	0.03011 89783	46 53	0.82801 32373	46
0.70702 13033	1.01579 49474	0.03015 36896	45 53	0.81001 29496	45
A(r)	D(r)	E(r)	ϕ	Fϕ	r

K = 1.6489952185, K′ = 2.3087867982, E = 1.4981149284, E′ = 1.1638279045,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01832 21691	1	3	0.00167 60815	1.00001 53505	0.01744 18501
2	0.03664 43382	2	6	0.00334 99667	1.00006 14074	0.03487 84245
3	0.05496 65073	3	9	0.00501 94629	1.00013 80964	0.05230 44041
4	0.07328 86764	4	12	0.00668 23842	1.00024 53303	0.06971 45088
5	0.09161 08455	5	15	0.00833 65551	1.00038 29783	0.08710 34544
6	0.10993 30145	6	18	0.00997 08130	1.00055 08728	0.10446 59627
7	0.12825 51836	7	21	0.01161 00163	1.00074 88602	0.12179 67635
8	0.14657 73527	8	24	0.01322 50382	1.00097 65463	0.13909 05958
9	0.16489 95218	9	26	0.01482 27797	1.00123 38067	0.15634 22068
10	0.18322 16909	10	29	0.01640 11677	1.00152 02770	0.17354 63660
11	0.20154 38600	11	32	0.01795 81596	1.00183 56081	0.19069 78446
12	0.21986 60291	12	35	0.01949 17458	1.00217 91150	0.20779 14445
13	0.23818 81982	13	37	0.02099 99543	1.00255 12815	0.22482 19184
14	0.25651 03673	14	40	0.02248 08485	1.00295 07519	0.24178 42052
15	0.27483 25364	15	43	0.02393 25396	1.00337 73404	0.25867 30615
16	0.29315 47055	16	45	0.02535 31708	1.00383 05272	0.27548 33938
17	0.31147 68746	17	48	0.02674 06700	1.00430 97603	0.29221 00649
18	0.32979 90437	18	50	0.02809 41600	1.00481 44557	0.30884 80221
19	0.34812 12128	19	53	0.02941 10555	1.00534 39986	0.32539 21091
20	0.36644 33819	20	56	0.03069 00118	1.00589 77438	0.34183 73073
21	0.38476 55510	21	57	0.03192 94445	1.00647 50167	0.35817 91274
22	0.40308 77201	22	59	0.03312 78272	1.00707 51140	0.37441 19097
23	0.42140 98892	24	1	0.03428 36045	1.00769 73046	0.39053 09808
24	0.43973 28582	25	3	0.03539 50434	1.00834 08404	0.40653 14452
25	0.45805 42273	26	5	0.03646 23352	1.00900 49074	0.42240 84004
26	0.47637 63964	27	7	0.03748 24070	1.00968 87266	0.43815 70635
27	0.49469 85655	28	9	0.03845 49232	1.01039 14518	0.45377 26140
28	0.51302 07346	29	11	0.03937 84764	1.01111 22358	0.46925 03045
29	0.53134 29037	30	12	0.04025 20886	1.01185 01016	0.48458 54241
30	0.54966 50728	31	14	0.04107 47627	1.01260 44231	0.49977 32999
31	0.56798 72419	32	15	0.04184 55726	1.01337 40114	0.51480 93092
32	0.58630 94110	33	16	0.04256 36643	1.01415 80186	0.52968 88704
33	0.60463 15801	34	18	0.04322 82564	1.01495 54890	0.54440 74492
34	0.62295 37492	35	19	0.04383 86406	1.01576 54545	0.55896 05660
35	0.64127 59183	36	20	0.04439 41821	1.01658 69227	0.57334 37664
36	0.65959 80874	37	21	0.04489 43196	1.01741 88967	0.58755 26819
37	0.67792 02565	38	22	0.04533 85685	1.01826 03617	0.60158 29737
38	0.69624 24256	39	23	0.04572 65058	1.01911 02927	0.61543 03611
39	0.71456 45947	40	23	0.04605 78000	1.01996 76540	0.62909 06189
40	0.73288 67638	41	23	0.04633 21809	1.02083 14013	0.64255 95777
41	0.75120 89328	42	24	0.04654 94543	1.02170 04820	0.65583 31258
42	0.76953 11019	43	24	0.04670 94981	1.02257 38374	0.66890 72089
43	0.78785 32710	44	24	0.04681 22622	1.02345 04035	0.68177 78347
44	0.80617 54401	45	24	0.04685 77678	1.02432 91132	0.69444 10704
45	0.82449 76092	46	24	0.04684 61065	1.02520 88930	0.70689 30463
90−r	Fψ	ψ		G(r)	C(r)	B(r)

TABLE $\theta = 25°$

$q = 0.012294560527181$, $\Theta 0 = 0.975410924642$, $HK = 0.666076159327$

B(r)	C(r)	G(r)	ψ		Fψ	90-r
1.00000 00000	1.05041 79735	0.00000 00000	90°	0′	1.64899 52185	90
0.99984 75111	1.05040 26107	0.00159 57045	89	3	1.63067 30494	89
0.99939 09912	1.05035 65652	0.00318 96046	88	6	1.61235 08803	88
0.99862 78812	1.05027 98750	0.00477 98977	87	9	1.59402 87112	87
0.99756 11158	1.05017 26395	0.00636 47840	86	12	1.57570 65421	86
0.99619 01245	1.05003 49805	0.00794 24686	85	15	1.55738 43730	85
0.99451 53203	1.04986 70026	0.00951 11627	84	17	1.53906 22039	84
0.99253 72400	1.04966 91533	0.01106 90855	83	20	1.52074 00348	83
0.99025 64734	1.04944 14139	0.01261 44653	82	23	1.50241 78657	82
0.98767 37287	1.04918 41480	0.01414 55416	81	26	1.48409 56966	81
0.98478 08010	1.04889 70746	0.01566 05663	80	29	1.46577 35275	80
0.98160 55779	1.04858 23391	0.01715 78054	79	31	1.44745 13584	79
0.97812 20405	1.04823 85265	0.01863 55407	78	34	1.42912 91893	78
0.97434 02570	1.04786 66550	0.02009 20712	77	37	1.41080 70202	77
0.97026 13902	1.04746 71803	0.02152 57149	76	39	1.39248 48511	76
0.96588 07101	1.04704 05862	0.02293 48102	75	42	1.37416 26821	75
0.96121 75452	1.04658 73036	0.02431 77177	74	44	1.35584 05130	74
0.95625 53477	1.04610 81546	0.02567 28218	73	47	1.33751 83439	73
0.95100 16139	1.04560 34540	0.02699 85322	72	49	1.31919 61748	72
0.94545 79803	1.04507 39038	0.02829 32857	71	52	1.30087 40057	71
0.93962 61680	1.04452 01522	0.02955 55477	70	54	1.28255 18366	70
0.93350 79411	1.04394 28728	0.03078 38140	69	56	1.26422 96675	69
0.92710 51976	1.04334 27090	0.03197 66123	68	58	1.24590 74984	68
0.92041 98038	1.04272 08710	0.03313 25038	68	0	1.22758 53293	67
0.91345 40932	1.04207 70306	0.03425 00853	67	2	1.20926 31602	66
0.90620 09299	1.04141 20561	0.03532 79902	66	4	1.19094 09911	65
0.89868 06404	1.04072 91305	0.03636 48907	65	6	1.17261 88220	64
0.89089 88058	1.04002 63060	0.03735 94992	64	8	1.15429 66529	63
0.88282 99477	1.03930 56088	0.03831 05700	63	10	1.13597 44838	62
0.87449 51326	1.03856 76470	0.03921 69099	62	11	1.11765 23147	61
0.86589 45184	1.03781 34098	0.04007 73349	61	13	1.09933 01456	60
0.85702 95411	1.03704 38161	0.04089 07619	60	14	1.08100 79765	59
0.84790 41300	1.03625 98035	0.04165 61200	59	16	1.06268 58075	58
0.83852 01741	1.03546 23472	0.04237 23976	58	17	1.04436 36384	57
0.82888 05840	1.03465 23588	0.04303 86345	57	18	1.02604 14693	56
0.81898 91269	1.03383 08852	0.04365 39236	56	19	1.00771 93002	55
0.80884 80221	1.03299 89073	0.04421 74127	55	20	0.98939 71311	54
0.79846 06482	1.03215 74380	0.04472 83056	54	21	0.97107 49620	53
0.78783 01874	1.03130 75044	0.04518 58637	53	22	0.95275 27929	52
0.77698 98986	1.03045 01401	0.04558 94076	52	22	0.93443 06238	51
0.76588 31015	1.02958 63985	0.04593 83183	51	23	0.91610 84547	50
0.75451 32083	1.02871 73077	0.04623 20386	50	24	0.89778 62856	49
0.74294 30775	1.02784 39507	0.04647 00744	49	24	0.87946 41165	48
0.73114 80583	1.02696 73835	0.04665 19961	48	24	0.86114 19474	47
0.71912 09561	1.02608 86741	0.04677 74393	47	24	0.84281 97783	46
0.70689 30463	1.02520 88930	0.04684 61065	46	24	0.82449 76092	45

K = 1.6857503548, K′ = 2.1565156475, E = 1.4674622093 E′ = 1.211056028,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01873 05595	1 4	0.00242 48763	1.00002 27125	0.01742 98716
2	0.03746 11190	2 9	0.00484 64683	1.00009 08222	0.03485 44751
3	0.05619 16785	3 13	0.00726 14977	1.00020 42462	0.05226 85438
4	0.07492 22380	4 18	0.00966 66975	1.00036 28463	0.06966 68140
5	0.09365 27975	5 22	0.01205 88178	1.00056 64204	0.08704 40267
6	0.11238 33570	6 26	0.01443 46319	1.00081 47472	0.10439 40285
7	0.13111 39165	7 30	0.01679 09412	1.00110 74975	0.12171 42736
8	0.14984 44760	8 35	0.01912 45813	1.00144 43235	0.13899 68254
9	0.16857 50355	9 39	0.02143 24269	1.00182 48148	0.15623 73574
10	0.18730 55950	10 43	0.02371 13976	1.00224 85070	0.17343 06551
11	0.20603 61545	11 47	0.02595 84626	1.00271 48868	0.19057 15175
12	0.22476 67140	12 51	0.02817 06450	1.00322 33830	0.20765 47584
13	0.24349 72734	13 55	0.03034 50312	1.00377 33773	0.22467 52081
14	0.26222 78329	14 59	0.03247 87664	1.00436 41996	0.24162 77146
15	0.28095 83924	16 3	0.03456 90685	1.00499 51300	0.25850 71451
16	0.29968 89519	17 6	0.03661 32272	1.00566 54000	0.27530 83886
17	0.31841 95114	18 10	0.03860 86097	1.00637 41929	0.29202 63549
18	0.33715 00709	19 14	0.04055 26642	1.00712 06453	0.30865 59785
19	0.35588 06304	20 17	0.04244 29236	1.00790 38477	0.32519 22190
20	0.37461 11899	21 20	0.04427 70092	1.00872 28461	0.34163 00625
21	0.39334 17494	22 23	0.04605 26335	1.00957 66426	0.35796 45236
22	0.41207 23089	23 27	0.04776 76034	1.01046 41971	0.37419 06461
23	0.43080 28684	24 30	0.04941 98229	1.01138 44282	0.39030 35051
24	0.44953 34279	25 33	0.05100 72958	1.01233 62150	0.40629 82084
25	0.46826 39874	26 36	0.05252 81275	1.01331 83078	0.42216 98975
26	0.48699 45469	27 38	0.05398 05273	1.01432 97800	0.43791 37495
27	0.50572 51064	28 41	0.05536 28100	1.01536 91295	0.45352 49782
28	0.52445 56659	29 43	0.05667 33976	1.01643 51800	0.46899 88358
29	0.54318 62254	30 46	0.05791 08204	1.01752 66329	0.48433 06142
30	0.56191 67849	31 48	0.05907 37181	1.01864 21583	0.49951 56464
31	0.58064 73444	32 50	0.06016 08407	1.01978 03072	0.51454 93080
32	0.59937 79039	33 52	0.06117 10486	1.02093 99629	0.52942 70185
33	0.61810 84634	34 54	0.06210 33138	1.02211 94428	0.54414 42428
34	0.63683 90229	35 55	0.06295 67191	1.02331 73997	0.55869 64925
35	0.65556 95824	36 56	0.06373 04587	1.02453 23743	0.57307 93274
36	0.67430 01419	37 58	0.06442 38375	1.02576 28863	0.58728 83566
37	0.69303 07014	38 59	0.06503 62710	1.02700 74365	0.60131 92403
38	0.71176 12609	40 0	0.06556 72843	1.02826 45087	0.61516 76907
39	0.73049 18204	41 1	0.06601 65112	1.02953 25714	0.62882 94738
40	0.74922 23799	42 2	0.06638 36938	1.03081 00797	0.64230 04103
41	0.76795 29394	43 3	0.06666 86806	1.03209 54771	0.65557 63772
42	0.78668 34989	44 3	0.06687 14255	1.03338 71976	0.66865 33089
43	0.80541 40584	45 3	0.06699 19865	1.03468 36674	0.68152 71988
44	0.82414 46179	46 4	0.06703 05237	1.03598 33070	0.69419 41003
45	0.84287 51774	47 3	0.06698 72981	1.03728 45330	0.70665 01282
90−r	Fψ	ψ	G(r)	C(r)	B(r)

K = 1.7312451757, K′ = 2.0347153122, E = 1.4322909693, E′ = 1.2586796248,

r	$F\phi$	ϕ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01923 60575	1 6	0.00332 09329	1.00003 10451	0.01740 91115
2	0.03847 21150	2 12	0.00663 71847	1.00012 77415	0.03481 29991
3	0.05770 81725	3 18	0.00994 40836	1.00028 72724	0.05220 64103
4	0.07694 42300	4 24	0.01323 69759	1.00051 03430	0.06958 42154
5	0.09618 02875	5 30	0.01651 12357	1.00079 66833	0.08694 11086
6	0.11541 63450	6 36	0.01976 22733	1.00114 59427	0.10427 19100
7	0.13465 24025	7 42	0.02298 55416	1.00155 76068	0.12157 14162
8	0.15388 84600	8 48	0.02617 65504	1.00203 14420	0.13883 44322
9	0.17312 45176	9 54	0.02933 08900	1.00256 66050	0.15605 57726
10	0.19236 05751	11 0	0.03244 41707	1.00316 25308	0.17323 02632
11	0.21159 66326	12 5	0.03551 21508	1.00381 84944	0.19035 27418
12	0.23083 26901	13 11	0.03853 06122	1.00453 30968	0.20741 80603
13	0.25006 87476	14 16	0.04149 54668	1.00530 72668	0.22442 10857
14	0.26930 48051	15 22	0.04440 27192	1.00613 82630	0.24135 67013
15	0.28854 08626	16 27	0.04724 84818	1.00702 56701	0.25821 98088
16	0.30777 69201	17 32	0.05002 89819	1.00796 84103	0.27500 53388
17	0.32701 29776	18 37	0.05274 05071	1.00896 53440	0.29170 82026
18	0.34624 90351	19 42	0.05537 97118	1.01001 52268	0.30832 33939
19	0.36548 50926	20 47	0.05794 30217	1.01111 68699	0.32484 58897
20	0.38472 11501	21 52	0.06042 72392	1.01226 87413	0.34127 07019
21	0.40395 72077	22 56	0.06282 92476	1.01346 96177	0.35759 28687
22	0.42319 32652	24 0	0.06514 60751	1.01471 79763	0.37380 74559
23	0.44242 93227	25 5	0.06737 48988	1.01601 22964	0.38990 95585
24	0.46166 53802	26 9	0.06951 30473	1.01735 10012	0.40589 43010
25	0.48090 14377	27 13	0.07155 80036	1.01873 24599	0.42175 68435
26	0.50013 74952	28 16	0.07350 74079	1.02015 49897	0.43749 23737
27	0.51937 35527	29 20	0.07535 90588	1.02161 68576	0.45309 61170
28	0.53860 96102	30 23	0.07711 09151	1.02311 62828	0.46856 33475
29	0.55784 56677	31 27	0.07876 10969	1.02465 14386	0.48388 93414
30	0.57708 17252	32 30	0.08030 78862	1.02622 01548	0.49906 94371
31	0.59631 77827	33 32	0.08174 97274	1.02782 14201	0.51409 90330
32	0.61555 38402	34 35	0.08308 52267	1.02945 23841	0.52897 35486
33	0.63478 98977	35 37	0.08431 31523	1.03111 13599	0.54368 84170
34	0.65402 59552	36 40	0.08543 24331	1.03279 63263	0.55823 01784
35	0.67326 20128	37 42	0.08644 21880	1.03450 52308	0.57262 13672
36	0.69249 80703	38 43	0.08734 15741	1.03623 59914	0.58683 05928
37	0.71173 41278	39 45	0.08813 00853	1.03798 69996	0.60086 25017
38	0.73097 01853	40 46	0.08880 72502	1.03975 40228	0.61471 27930
39	0.75020 62428	41 48	0.08937 37798	1.04153 82068	0.62837 72177
40	0.76944 23003	42 49	0.08982 65352	1.04333 80787	0.64185 15702
41	0.78867 83578	43 49	0.09016 85246	1.04514 30495	0.65513 17355
42	0.80791 44153	44 50	0.09039 89009	1.04695 99164	0.66821 35909
43	0.82715 04728	45 50	0.09051 79579	1.04878 34660	0.68109 31428
44	0.84638 65303	46 51	0.09052 61280	1.05061 14768	0.69376 69046
45	0.86562 25878	47 51	0.09042 39779	1.05244 17208	0.70622 94378
90 – r	$F\psi$	ψ	G(r)	C(r)	B(r)

$q = 0.024015062623081$, $OO = 0.9501700466$, $HK = 0.7950876364$

B(r)	C(r)	G(r)	ψ		Fψ	90−r
1.00000 00000	1.10488 66850	0.00000 00000	90°	0′	1.73124 51757	90
0.99984 60394	1.10485 47360	0.00300 62320	89	6	1.71200 91181	89
0.99938 78065	1.10475 89287	0.00600 93218	88	12	1.69277 30606	88
0.99862 27471	1.10459 93781	0.00900 61288	87	17	1.67353 70031	87
0.99755 20048	1.10437 62705	0.01199 35156	86	23	1.65430 09456	86
0.99617 59200	1.10408 99048	0.01496 83495	85	29	1.63506 48881	85
0.99449 49395	1.10374 06020	0.01792 75043	84	35	1.61582 88306	84
0.99250 95707	1.10332 87096	0.02086 78620	83	40	1.59659 27731	83
0.99022 04719	1.10285 49965	0.02378 63141	82	46	1.57735 67156	82
0.98762 83615	1.10231 97711	0.02667 97640	81	51	1.55812 06581	81
0.98473 40633	1.10172 37756	0.02954 51270	80	57	1.53888 46006	80
0.98154 84966	1.10106 77362	0.03237 93372	80	2	1.51964 85431	79
0.97804 26763	1.10035 24524	0.03517 93404	79	8	1.50041 24856	78
0.97424 77117	1.09957 87057	0.03794 21046	78	13	1.48117 64281	77
0.97015 48673	1.09874 77080	0.04066 46178	77	19	1.46194 03706	76
0.96576 52012	1.09786 02047	0.04334 38907	76	24	1.44270 43130	75
0.96108 04640	1.09691 73640	0.04597 69592	75	29	1.42346 82555	74
0.95610 19028	1.09592 03375	0.04856 08861	74	34	1.40423 21980	73
0.95083 11516	1.09487 03382	0.05109 27637	73	38	1.38499 61405	72
0.94526 95796	1.09376 86463	0.05356 97161	72	43	1.36576 00830	71
0.93941 98461	1.09261 66042	0.05598 89014	71	48	1.34652 40255	70
0.93328 29095	1.09141 56150	0.05834 75147	70	52	1.32728 79680	69
0.92686 08817	1.09016 71440	0.06064 27902	69	56	1.30805 19105	68
0.92015 61173	1.08887 27107	0.06287 20041	69	1	1.28881 58530	67
0.91317 04228	1.08753 38030	0.06503 24775	68	5	1.26957 97955	66
0.90590 61067	1.08615 23221	0.06712 15792	67	9	1.25034 37380	65
0.89836 54996	1.08473 96815	0.06913 67285	66	12	1.23110 76805	64
0.89055 08145	1.08326 77048	0.07107 53988	65	16	1.21187 16230	63
0.88246 46805	1.08176 81732	0.07293 51200	64	19	1.19263 55655	62
0.87410 95824	1.08023 29140	0.07471 34824	63	23	1.17339 95080	61
0.86548 81427	1.07866 37078	0.07640 81398	62	26	1.15416 34504	60
0.85660 30670	1.07706 27365	0.07801 68127	61	29	1.13492 73929	59
0.84745 71408	1.07543 16809	0.07953 72924	60	31	1.11569 13354	58
0.83805 32400	1.07377 26184	0.08096 74440	59	34	1.09645 52779	57
0.82840 42745	1.07208 78705	0.08230 52102	58	36	1.07721 92204	56
0.81848 32073	1.07037 85902	0.08354 86152	57	39	1.05798 31629	55
0.80834 33933	1.06864 77599	0.08469 57684	56	41	1.03874 71054	54
0.79791 77333	1.06689 71884	0.08574 48680	55	43	1.01951 10479	53
0.78720 95015	1.06512 99086	0.08669 42053	54	44	1.00027 49904	52
0.77638 21045	1.06334 53750	0.08754 21680	53	46	0.98103 89329	51
0.76525 90201	1.06154 84606	0.08828 72448	52	48	0.96180 28754	50
0.75390 34961	1.05974 04548	0.08892 80287	51	49	0.94256 68179	49
0.74231 91490	1.05792 35605	0.08946 32214	50	49	0.92333 07604	48
0.73050 95727	1.05609 99913	0.08989 16370	49	50	0.90409 47028	47
0.71847 84273	1.05427 19690	0.09021 22056	48	50	0.88485 86453	46
0.70622 94378	1.05244 17208	0.09042 39779	47	51	0.85562 25878	45
A(r)	D(r)	E(r)	φ		Fφ	r

K = 1.7867691349, K′ = 1.9355810960, E = 1.3031402485, E′ = 1.3055300943,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.01985 29904	1 8	0.00437 25767	1.00004 34107	0.01737 52057
2	0.03970 59807	2 16	0.00873 86910	1.00017 35897	0.03474 53706
3	0.05955 89712	3 24	0.01309 18945	1.00039 03787	0.05210 51913
4	0.07941 19615	4 32	0.01742 57081	1.00069 35136	0.06944 95525
5	0.09926 49519	5 41	0.02173 39351	1.00108 26253	0.08677 33185
6	0.11911 79423	6 49	0.02601 00761	1.00155 72398	0.10407 13496
7	0.13897 09327	7 57	0.03024 79420	1.00211 67701	0.12133 85117
8	0.15882 39231	9 5	0.03444 13083	1.00276 05620	0.13856 96786
9	0.17867 69135	10 13	0.03858 42875	1.00348 78042	0.15575 97300
10	0.19852 99039	11 21	0.04267 07422	1.00429 76203	0.17290 35587
11	0.21838 28943	12 28	0.04669 48973	1.00518 90230	0.18999 60057
12	0.23823 58847	13 36	0.05065 10510	1.00616 09298	0.20703 21648
13	0.25808 88751	14 43	0.05453 36499	1.00721 21534	0.22400 67828
14	0.27794 18655	15 51	0.05833 72913	1.00834 14154	0.24091 48669
15	0.29779 48558	16 58	0.06205 67422	1.00954 73402	0.25775 13559
16	0.31764 78462	18 5	0.06568 69435	1.01082 84502	0.27451 14417
17	0.33750 08366	19 12	0.06922 30203	1.01218 32120	0.29118 95009
18	0.35735 38270	20 18	0.07266 02808	1.01360 99187	0.30778 11718
19	0.37720 68174	21 25	0.07599 42073	1.01510 60418	0.32428 12503
20	0.39705 98078	22 31	0.07922 06754	1.01667 23379	0.34068 48260
21	0.41691 27981	23 37	0.08233 54475	1.01830 42660	0.35698 69191
22	0.43676 57885	24 42	0.08533 47336	1.02000 07123	0.37318 27300
23	0.45661 87789	25 48	0.08821 49046	1.02175 96267	0.38926 72959
24	0.47647 17693	26 53	0.09097 25564	1.02357 88016	0.40523 58014
25	0.49632 47597	27 59	0.09360 45123	1.02545 62012	0.42108 34203
26	0.51617 77501	29 4	0.09610 78252	1.02738 93589	0.43680 53024
27	0.53603 07405	30 8	0.09847 07702	1.02937 59801	0.45239 69341
28	0.55588 37309	31 13	0.10071 78005	1.03141 36450	0.46785 33318
29	0.57573 67212	32 17	0.10281 99075	1.03349 98717	0.48316 98948
30	0.59558 97116	33 22	0.10478 38101	1.03563 21191	0.49834 19088
31	0.61544 27020	34 25	0.10660 78692	1.03780 77899	0.51336 49360
32	0.63529 56924	35 28	0.10829 03444	1.04002 42340	0.52823 42166
33	0.65514 86828	36 31	0.10983 00821	1.04227 87515	0.54294 52702
34	0.67500 16732	37 34	0.11122 59132	1.04456 85961	0.55749 35973
35	0.69485 46636	38 37	0.11247 69491	1.04689 09786	0.57187 47405
36	0.71470 76540	39 39	0.11358 25187	1.04924 30699	0.58608 42864
37	0.73456 06443	40 41	0.11454 21645	1.05162 20047	0.60011 78668
38	0.75441 36347	41 42	0.11535 50375	1.05402 48851	0.61397 11899
39	0.77426 66251	42 44	0.11602 28032	1.05644 87839	0.62763 98902
40	0.79411 96155	43 46	0.11654 40861	1.05889 07481	0.64111 98356
41	0.81397 26059	44 46	0.11691 95649	1.06134 78029	0.65440 68220
42	0.83382 55963	45 47	0.11714 98662	1.06381 69550	0.66749 67282
43	0.85367 85867	46 47	0.11723 57096	1.06629 51962	0.68038 54871
44	0.87353 15771	47 48	0.11717 79914	1.06877 95074	0.69306 90869
45	0.89338 45674	48 48	0.11697 77784	1.07126 68617	0.70554 35725
90−r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.0332052560095677$, $\Theta_0 = 0.0334719360$, $HK = 0.8550825245$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	1.14254 42177	0.00000 00000	90° 0′	1.78676 91349	90
0.99984 63487	1.14250 07942	0.00382 84907	89 8	1.76691 61445	89
0.99938 54451	1.14237 05760	0.00765 31872	88 15	1.74706 31541	88
0.99861 71408	1.14215 37243	0.01147 02963	87 23	1.72721 01637	87
0.99754 25880	1.14185 05008	0.01527 60269	86 30	1.70735 71733	86
0.99616 12401	1.14146 12760	0.01906 65913	85 38	1.68750 41829	85
0.99447 38506	1.14098 65243	0.02283 82057	84 46	1.66765 11926	84
0.99248 00741	1.14042 68243	0.02658 70918	83 53	1.64779 82022	83
0.99018 39628	1.13978 28384	0.03030 94781	83 1	1.62794 52118	82
0.98758 14720	1.13905 54113	0.03400 16000	82 8	1.60809 22214	81
0.98467 64560	1.13824 53008	0.03765 97054	81 16	1.58823 92310	80
0.98146 01652	1.13735 37241	0.04128 00477	80 23	1.56838 62406	79
0.97796 00500	1.13638 15521	0.04485 88958	79 30	1.54853 32502	78
0.97415 29616	1.13533 00176	0.04839 25314	78 37	1.52868 02598	77
0.97004 46432	1.13420 01894	0.05187 72514	77 44	1.50882 72694	76
0.96564 07380	1.13299 42530	0.05530 93702	76 51	1.48897 42791	75
0.96094 82806	1.13171 28116	0.05868 52206	75 57	1.46912 12887	74
0.95594 33213	1.13035 77242	0.06200 11573	75 4	1.44926 82983	73
0.95066 49710	1.12893 06133	0.06525 35577	74 10	1.42941 53079	72
0.94507 54604	1.12743 33082	0.06843 88251	73 17	1.40956 23175	71
0.93920 66032	1.12586 75138	0.07155 33910	72 23	1.38970 93271	70
0.93305 03052	1.12423 52584	0.07459 37177	71 29	1.36985 63367	69
0.92660 88744	1.12253 84111	0.07755 63011	70 34	1.35000 33463	68
0.91988 34013	1.12077 91607	0.08043 76736	69 40	1.33015 03560	67
0.91287 72478	1.11895 95604	0.08323 44077	68 45	1.31029 73656	66
0.90559 20807	1.11708 18582	0.08594 31188	67 51	1.29044 43752	65
0.89804 04745	1.11514 84422	0.08856 04692	66 56	1.27059 13848	64
0.89020 45608	1.11316 13660	0.09108 31714	66 0	1.25073 83944	63
0.88209 71618	1.11112 33590	0.09350 79023	65 5	1.23088 54040	62
0.87371 07901	1.10903 67086	0.09583 17573	64 9	1.21103 24136	61
0.86506 81367	1.10690 44270	0.09805 13845	63 14	1.19117 94233	60
0.85616 10781	1.10472 82468	0.10016 37491	62 18	1.17132 64329	59
0.84699 51503	1.10251 15061	0.10216 59383	61 21	1.15147 34425	58
0.83757 06220	1.10025 67080	0.10405 50557	60 25	1.13162 04521	57
0.82789 13761	1.09796 63991	0.10582 82770	59 28	1.11176 74617	56
0.81796 03020	1.09564 40724	0.10748 28746	58 32	1.09191 44713	55
0.80778 11084	1.09329 16586	0.10901 62132	57 34	1.07206 14809	54
0.79735 66601	1.09091 28160	0.11042 57553	56 37	1.05220 84905	53
0.78669 01452	1.08850 01825	0.11170 90668	55 39	1.03235 55001	52
0.77578 81173	1.08608 33032	0.11286 38228,	54 42	1.01250 25098	51
0.76464 80133	1.08364 32047	0.11388 78137	53 44	0.99264 95194	50
0.75327 33376	1.08118 01237	0.11477 89511	52 45	0.97279 65290	49
0.74167 36742	1.07871 68830	0.11553 52736	51 46	0.95294 35386	48
0.72984 96728	1.07623 88782	0.11615 49535	50 46	0.93309 05482	47
0.71780 80468	1.07375 42288	0.11663 63025	49 47	0.91323 75578	46
0.70554 38725	1.07126 68617	0.11697 77784	48 48	0.89338 45674	45
A(r)	D(r)	E(r)	ϕ	Fϕ	r

K = K′ = 1.8540746773, E = E′ = 1.3506438810,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02060 08297	1 11	0.00559 22185	1.00005 79114	0.01732 23240
2	0.04120 16595	2 22	0.01117 86095	1.00023 03753	0.03463 96092
3	0.06180 24892	3 32	0.01674 17286	1.00051 80814	0.05194 68175
4	0.08240 33190	4 43	0.02228 16313	1.00092 03706	0.06923 89126
5	0.10300 41487	5 54	0.02778 68124	1.00143 67802	0.08651 08611
6	0.12360 49785	7 4	0.03324 87400	1.00206 66547	0.10375 76329
7	0.14420 58082	8 15	0.03865 90273	1.00280 92404	0.12097 42023
8	0.16480 66380	9 25	0.04400 93780	1.00366 30213	0.13815 55494
9	0.18540 74677	10 36	0.04929 16089	1.00462 87096	0.15529 66598
10	0.20600 82975	11 46	0.05449 79400	1.00570 35068	0.17239 25270
11	0.22660 91272	12 56	0.05962 01166	1.00688 69247	0.18943 81584
12	0.24720 99570	14 6	0.06465 13306	1.00817 63613	0.20642 85403
13	0.26781 07867	15 15	0.06958 30331	1.00957 13091	0.22335 87204
14	0.28841 16165	16 25	0.07441 05120	1.01107 02098	0.24022 37330
15	0.30901 24462	17 34	0.07912 41078	1.01267 06202	0.25701 86008
16	0.32961 32760	18 43	0.08371 09297	1.01437 09030	0.27373 83803
17	0.35021 41057	19 52	0.08818 26301	1.01616 88793	0.29037 81091
18	0.37081 49355	21 1	0.09252 84012	1.01806 24909	0.30693 30262
19	0.39141 57652	22 9	0.09672 53055	1.02004 91191	0.32339 80622
20	0.41201 65950	23 17	0.10078 25724	1.02212 67193	0.33976 83967
21	0.43261 74247	24 25	0.10469 18308	1.02429 25769	0.35603 91671
22	0.45321 82545	25 33	0.10844 83155	1.02654 40853	0.37220 55308
23	0.47381 90842	26 40	0.11204 76417	1.02887 86045	0.38826 26656
24	0.49441 99139	27 47	0.11548 55640	1.03129 29803	0.40420 87714
25	0.51502 07437	28 54	0.11875 82813	1.03378 46098	0.42003 08711
26	0.53562 15734	30 0	0.12186 22028	1.03635 03103	0.43573 08120
27	0.55622 24032	31 6	0.12479 44425	1.03898 69080	0.45130 43670
28	0.57682 32329	32 12	0.12755 18730	1.04169 14251	0.46674 27359
29	0.59742 40627	33 17	0.13013 20787	1.04446 04298	0.48204 48408
30	0.61802 48924	34 22	0.13253 28601	1.04729 03371	0.49720 40872
31	0.63862 57222	35 27	0.13475 24113	1.05017 79739	0.51221 66850
32	0.65922 65519	36 32	0.13678 86728	1.05311 97829	0.52707 77648
33	0.67982 73817	37 36	0.13864 11903	1.05611 20812	0.54178 28331
34	0.70042 82114	38 39	0.14030 86741	1.05915 11400	0.55632 73509
35	0.72102 90412	39 43	0.14179 07157	1.06223 37824	0.57070 68507
36	0.74162 98709	40 46	0.14308 61509	1.06535 86407	0.58491 69601
37	0.76223 07007	41 48	0.14419 18891	1.06851 31242	0.59895 31001
38	0.78283 15304	42 51	0.14511 90660	1.07170 25604	0.61281 10868
39	0.80343 23602	43 54	0.14585 76840	1.07491 97640	0.62648 65530
40	0.82403 31899	44 54	0.14641 09671	1.07816 10137	0.63997 82333
41	0.84463 40197	45 55	0.14678 03964	1.08142 24130	0.65327 20930
42	0.86523 48494	46 56	0.14696 71583	1.08469 96916	0.66637 83880
43	0.88583 56792	47 57	0.14697 20041	1.08798 91821	0.67927 88628
44	0.90643 65089	48 57	0.14679 85468	1.09128 66967	0.69197 83814
45	0.92703 73387	49 57	0.14644 66604	1.09458 82586	0.70447 07318

$q = e^{-\pi} = 0.04321391826377$, $\Theta 0 = 0.9135791382$, $HK = 0.9135791382$

B(r)	C(r)	G(r)	ψ		Fψ	90−r
1.00000 00000	1.18920 71150	0.00000 00000	90°	0′	1.85407 46773	90
0.99984 54246	1.18914 94665	0.00470 60108	89	10	1.83347 38476	89
0.99938 17514	1.18897 65912	0.00940 76502	88	20	1.81287 30178	88
0.99860 91406	1.18868 87000	0.01410 05467	87	30	1.79227 21881	87
0.99752 78584	1.18828 61440	0.01878 03289	86	40	1.77167 13583	86
0.99613 82775	1.18776 94140	0.02344 26255	85	49	1.75107 05286	85
0.99444 08767	1.18713 91403	0.02808 30653	84	59	1.73046 96988	84
0.99243 62407	1.18639 60914	0.03269 72774	84	9	1.70986 88691	83
0.99012 50593	1.18554 11736	0.03728 08916	83	18	1.68926 80393	82
0.98750 81276	1.18457 54293	0.04182 95382	82	28	1.66866 72096	81
0.98458 63450	1.18350 00303	0.04633 88487	81	37	1.64806 63798	80
0.98136 07151	1.18231 63059	0.05080 44575	80	47	1.62746 55501	79
0.97783 23446	1.18102 56817	0.05522 19994	79	56	1.60686 47203	78
0.97400 24430	1.17962 97376	0.05958 71130	79	5	1.58626 38906	77
0.96987 23216	1.17813 01756	0.06389 54439	78	14	1.56566 30608	76
0.96544 33929	1.17652 88244	0.06814 26379	77	23	1.54506 22311	75
0.96071 71696	1.17482 76366	0.07232 43506	76	32	1.52446 14013	74
0.95569 52639	1.17302 86866	0.07643 62449	75	40	1.50386 05716	73
0.95037 93863	1.17113 41680	0.08047 39033	74	48	1.48325 97418	72
0.94477 13447	1.16914 63907	0.08443 32799	73	57	1.46265 89121	71
0.93887 30433	1.16706 77783	0.08830 98027	73	5	1.44205 80823	70
0.93268 64814	1.16490 08653	0.09208 92756	72	13	1.42145 72526	69
0.92621 37526	1.16264 82037	0.09579 74315	71	20	1.40085 64228	68
0.91945 70430	1.16031 28007	0.09940 00252	70	27	1.38025 55931	67
0.91241 86305	1.15789 72608	0.10290 28362	69	34	1.35965 47634	66
0.90510 08831	1.15540 45020	0.10630 16727	68	41	1.33905 39336	65
0.89750 62579	1.15283 78419	0.10959 23752	67	48	1.31845 31039	64
0.88963 72905	1.15020 01398	0.11277 08206	66	54	1.29785 22741	63
0.88149 66386	1.14749 47011	0.11583 29266	66	0	1.27725 14444	62
0.87308 69906	1.14472 48239	0.11877 46567	65	6	1.25665 06146	61
0.86441 11542	1.14189 38846	0.12159 20252	64	11	1.23604 97849	60
0.85547 20099	1.13900 53339	0.12428 11025	63	16	1.21544 89551	59
0.84627 25182	1.13606 26928	0.12683 80211	62	21	1.19484 81254	58
0.83681 57184	1.13306 95480	0.12925 89815	61	26	1.17424 72956	57
0.82710 47269	1.13002 95477	0.13154 02588	60	30	1.15364 64659	56
0.81714 27355	1.12694 63970	0.13367 82099	59	34	1.13304 56361	55
0.80693 30009	1.12382 38537	0.13566 92789	58	38	1.11244 48064	54
0.79647 88881	1.12066 57231	0.13751 00077	57	42	1.09184 39766	53
0.78578 37785	1.11747 58542	0.13919 70407	56	45	1.07124 31469	52
0.77485 11587	1.11425 81342	0.14072 71344	55	47	1.05064 23171	51
0.76368 45735	1.11101 64844	0.14209 71663	54	50	1.03004 14874	50
0.75228 70332	1.10775 48548	0.14330 41415	53	52	1.00944 06576	49
0.74066 40121	1.10447 72199	0.14434 52037	52	53	0.98883 98279	48
0.72881 74469	1.10118 75735	0.14521 76436	51	55	0.96823 89981	47
0.71675 17348	1.09788 99237	0.14591 89078	50	56	0.94763 81684	46
0.70447 07318	1.09458 82886	0.14644 66094	49	57	0.92703 73387	45

K = 1.9355810960, K′ = 1.7867691349, E = 1.3055390043, E′ = 1.3931402485,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02150 64566	1 14	0.00699 85212	1.00007 52700	0.01724 17831
2	0.04301 29132	2 28	0.01398 53703	1.00030 09884	0.03447 86990
3	0.06451 93699	3 41	0.02094 89334	1.00067 68800	0.05170 58810
4	0.08602 58265	4 55	0.02787 76288	1.00120 24903	0.06891 84030
5	0.10753 22831	6 9	0.03476 00806	1.00187 71775	0.08611 15805
6	0.12903 87397	7 22	0.04158 42717	1.00270 01222	0.10328 03705
7	0.15054 51963	8 36	0.04834 06320	1.00367 03237	0.12041 99725
8	0.17205 16530	9 49	0.05501 67694	1.00478 66023	0.13752 55283
9	0.19355 81096	11 3	0.06160 24003	1.00604 70005	0.15459 21831
10	0.21506 45662	12 16	0.06808 70479	1.00745 17850	0.17161 50856
11	0.23657 10228	13 28	0.07446 05194	1.00899 74482	0.18858 93888
12	0.25807 74795	14 41	0.08071 20320	1.01068 27105	0.20551 02505
13	0.27958 39361	15 53	0.08683 47367	1.01250 55225	0.22237 28335
14	0.30109 03927	17 6	0.09281 67403	1.01446 30673	0.23917 23067
15	0.32259 68493	18 18	0.09865 01256	1.01655 47645	0.25590 38457
16	0.34410 33059	19 29	0.10432 64604	1.01877 62678	0.27256 20330
17	0.36560 97626	20 40	0.10983 77503	1.02112 84784	0.28914 38591
18	0.38711 62192	21 51	0.11517 64068	1.02359 95379	0.30564 27234
19	0.40862 26758	23 2	0.12033 52604	1.02619 54370	0.32205 44344
20	0.43012 91324	24 13	0.12530 76146	1.02891 00179	0.33837 43110
21	0.45163 55891	25 22	0.13008 72182	1.03173 90787	0.35459 72832
22	0.47314 20457	26 31	0.13466 82799	1.03468 18764	0.37071 88930
23	0.49464 85023	27 41	0.13904 54724	1.03773 21323	0.38673 42053
24	0.51615 49589	28 50	0.14321 39340	1.04088 70352	0.40263 87589
25	0.53766 14155	29 59	0.14716 92687	1.04414 27466	0.41842 75678
26	0.55916 78722	31 6	0.15090 75443	1.04749 53052	0.43409 60218
27	0.58067 43288	32 14	0.15442 52802	1.05094 06315	0.44963 94381
28	0.60218 07854	33 21	0.15771 94871	1.05447 45329	0.46505 31522
29	0.62368 72420	34 29	0.16078 75703	1.05809 27090	0.48033 25191
30	0.64519 36987	35 36	0.16362 74123	1.06179 07501	0.49547 29148
31	0.66670 01553	36 41	0.16623 73178	1.06556 41737	0.51046 97376
32	0.68820 66119	37 46	0.16861 60131	1.06940 83686	0.52531 84091
33	0.70971 30685	38 51	0.17076 20341	1.07331 86617	0.54001 43761
34	0.73121 95251	39 56	0.17267 07142	1.07729 02929	0.55455 31110
35	0.75272 59818	41 1	0.17435 81713	1.08131 84270	0.56893 01177
36	0.77423 24384	42 4	0.17580 72036	1.08539 81601	0.58314 09242
37	0.79573 88950	43 7	0.17702 47258	1.08952 45247	0.59718 10935
38	0.81724 53516	44 9	0.17801 14536	1.09369 24065	0.61104 62201
39	0.83875 18083	45 12	0.17876 87890	1.09789 70801	0.62473 19335
40	0.86025 82649	46 15	0.17929 83544	1.10213 29153	0.63823 38991
41	0.88176 47215	47 15	0.17960 20675	1.10639 50831	0.65154 78204
42	0.90327 11781	48 16	0.17968 21252	1.11067 83124	0.66466 94406
43	0.92477 76347	49 16	0.17954 09878	1.11497 73861	0.67759 45449
44	0.94628 40914	50 17	0.17918 13641	1.11928 70673	0.69031 89618
45	0.96779 05480	51 17	0.17860 61952	1.12360 21058	0.70283 85652
90−r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.065019933698820$, $\Theta 0 = 0.8890784604$, $HK = 0.9715669451$

B(r)	C(r)	G(r)	ψ		Fψ	90−r
1.00000 00000	1.24728 65857	0.00000 00000	90°	0′	1.93558 10960	90
0.99984 40186	1.24721 12154	0.00561 92362	89	12	1.91407 46394	89
0.99937 61319	1.24698 51964	0.01123 36482	88	25	1.89256 81828	88
0.99859 65127	1.24660 88048	0.01683 84106	87	37	1.87106 17261	87
0.99750 54487	1.24608 24999	0.02242 89646	86	50	1.84955 52695	86
0.99610 33424	1.24540 69243	0.02799 96670	86	2	1.82804 88129	85
0.99439 07108	1.24458 29027	0.03354 64884	85	14	1.80654 23563	84
0.99236 81849	1.24361 14410	0.03906 43123	84	26	1.78503 58997	83
0.99003 65993	1.24249 37250	0.04454 82835	83	39	1.76352 94430	82
0.98739 65416	1.24123 11192	0.04999 35367	82	51	1.74202 29864	81
0.98444 92817	1.23982 51648	0.05539 51961	82	3	1.72051 65298	80
0.98119 57210	1.23827 75770	0.06074 83740	81	14	1.69901 00732	79
0.97763 71417	1.23659 02476	0.06604 81700	80	26	1.67750 36165	78
0.97377 48160	1.23476 52334	0.07128 96708	79	37	1.65599 71599	77
0.96961 01546	1.23280 47639	0.07646 79497	78	49	1.63449 07033	76
0.96514 46762	1.23071 12287	0.08157 80662	78	0	1.61298 42467	75
0.96038 00059	1.22848 71860	0.08661 50665	77	10	1.59147 77901	74
0.95531 78745	1.22613 53491	0.09157 39836	76	21	1.56997 13334	73
0.94996 01167	1.22365 85882	0.09644 98379	75	31	1.54846 48768	72
0.94430 86698	1.22105 99257	0.10123 76383	74	42	1.52695 84202	71
0.93836 55727	1.21834 25338	0.10593 23833	73	52	1.50545 19636	70
0.93213 20039	1.21550 97252	0.11052 90627	73	1	1.48394 55069	69
0.92561 30862	1.21256 49596	0.11502 26595	72	11	1.46243 90503	68
0.91880 84552	1.20951 18289	0.11940 81521	71	20	1.44093 25937	67
0.91172 00173	1.20635 40582	0.12368 05174	70	30	1.41942 61371	66
0.90435 38883	1.20309 54999	0.12783 47335	69	39	1.39791 96805	65
0.89670 88815	1.19974 01294	0.13186 57834	68	47	1.37641 32238	64
0.88878 94998	1.19629 20306	0.13576 86595	67	55	1.35490 67672	63
0.88059 82441	1.19275 54368	0.13953 83674	67	2	1.33340 03106	62
0.87213 79612	1.18913 46345	0.14316 99314	66	10	1.31189 38540	61
0.86341 16420	1.18543 40490	0.14665 83999	65	18	1.29038 73973	60
0.85442 23108	1.18165 81935	0.14999 88516	64	24	1.26888 09407	59
0.84517 31166	1.17781 16727	0.15318 64017	63	30	1.24737 44841	58
0.83566 72345	1.17389 91774	0.15621 62095	62	36	1.22586 80275	57
0.82590 79506	1.16992 54783	0.15908 34859	61	42	1.20436 15709	56
0.81589 86161	1.16589 54205	0.16178 35017	60	48	1.18285 51142	55
0.80564 26543	1.16181 39175	0.16431 15964	59	52	1.16134 86576	54
0.79514 35583	1.15768 59453	0.16666 31878	58	56	1.13984 22010	53
0.78440 48891	1.15351 65361	0.16883 37818	58	0	1.11833 57444	52
0.77343 02735	1.14931 07723	0.17081 89832	57	4	1.09682 92877	51
0.76222 34010	1.14507 37802	0.17261 45069	56	8	1.07532 28311	50
0.75078 80264	1.14081 07240	0.17421 61892	55	10	1.05381 63745	49
0.73912 79584	1.13652 67992	0.17562 00006	54	12	1.03230 99179	48
0.72724 70671	1.13222 72263	0.17682 20583	53	13	1.01080 34613	47
0.71514 92767	1.12791 72446	0.17781 86395	52	15	0.98929 70046	46
0.70283 85652	1.12360 21058	0.17860 61952	51	17	0.96779 05480	45
A(r)	D(r)	E(r)	φ		Fφ	r

K = 2.0347153122, K′ = 1.7312451757, E = 1.2580796248, E′ = 1.4322909093,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02260 79479	1 18	0.00862 00346	1.00009 74000	0.01712 13223
2	0.04521 58958	2 35	0.01722 45749	1.00038 97217	0.03423 80342
3	0.06782 38437	3 53	0.02579 81795	1.00087 64305	0.05134 55249
4	0.09043 17916	5 10	0.03432 55123	1.00155 69957	0.06843 91832
5	0.11303 97395	6 28	0.04279 13042	1.00243 05014	0.08551 43971
6	0.13564 76875	7 45	0.05118 08539	1.00349 61575	0.10256 65538
7	0.15825 56354	9 2	0.05947 91769	1.00475 24006	0.11959 10390
8	0.18086 35833	10 19	0.06767 19530	1.00619 77062	0.13658 32373
9	0.20347 15312	11 36	0.07574 51216	1.00783 05901	0.15353 85318
10	0.22607 94791	12 52	0.08368 50144	1.00964 88003	0.17045 23039
11	0.24868 74270	14 9	0.09147 83960	1.01165 02201	0.18731 09332
12	0.27129 53749	15 25	0.09911 25013	1.01383 24100	0.20413 67075
13	0.29390 33229	16 40	0.10657 50094	1.01619 27508	0.22089 82730
14	0.31651 12708	17 56	0.11385 43755	1.01872 83473	0.23759 97340
15	0.33911 92187	19 11	0.12093 02580	1.02143 61311	0.25423 65532
16	0.36172 71666	20 25	0.12781 91435	1.02431 28147	0.27080 41017
17	0.38433 51145	21 40	0.13448 40670	1.02735 49080	0.28729 77406
18	0.40694 30624	22 54	0.14092 46901	1.03055 87080	0.30371 28056
19	0.42955 10103	24 7	0.14713 23140	1.03392 03441	0.32004 48178
20	0.45215 89583	25 20	0.15300 88906	1.03743 56974	0.33628 89743
21	0.47476 69062	26 33	0.15881 70288	1.04110 05414	0.35244 07031
22	0.49737 48541	27 45	0.16427 99980	1.04491 03831	0.36849 53729
23	0.51998 28020	28 56	0.16948 17327	1.04886 06244	0.38444 83538
24	0.54259 07499	30 8	0.17441 68208	1.05294 64588	0.40029 50181
25	0.56519 86978	31 18	0.17908 05075	1.05716 20130	0.41603 07408
26	0.58780 66457	32 28	0.18346 86827	1.06150 48720	0.43165 09003
27	0.61041 45937	33 38	0.18757 78710	1.06596 70560	0.44715 08801
28	0.63302 25416	34 46	0.19140 52188	1.07054 40415	0.46253 60691
29	0.65563 04895	35 55	0.19494 84794	1.07523 02647	0.47777 18037
30	0.67823 84374	37 3	0.19820 50959	1.08002 00285	0.49288 30645
31	0.70084 63853	38 10	0.20117 66827	1.08490 75692	0.50785 08872
32	0.72345 43332	39 16	0.20386 00053	1.08988 67644	0.52268 69541
33	0.74606 22811	40 23	0.20625 59591	1.09498 17388	0.53736 93004
34	0.76867 02290	41 28	0.20836 50468	1.10009 62656	0.55180 93747
35	0.79127 81769	42 33	0.21018 82554	1.10531 40047	0.56627 26408
36	0.81388 61249	43 38	0.21172 70324	1.11059 88749	0.58048 45794
37	0.83649 40728	44 41	0.21298 32611	1.11594 41760	0.59453 06894
38	0.85910 20207	45 45	0.21395 02364	1.12134 34029	0.60840 64605
39	0.88170 99686	46 48	0.21465 76400	1.12679 02842	0.62210 75244
40	0.90431 79165	47 50	0.21508 15155	1.13227 78297	0.63562 93571
41	0.92692 58644	48 51	0.21523 42440	1.13779 95386	0.64896 75812
42	0.94953 38123	49 53	0.21511 05200	1.14334 86579	0.66211 78175
43	0.97214 17602	50 53	0.21474 13276	1.14891 84299	0.67507 57177
44	0.99474 97081	51 53	0.21410 39170	1.15450 20711	0.68783 69663
45	1.01735 76561	52 52	0.21321 17818	1.16009 27862	0.70039 72833
90-r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.060042299600032$, $\Theta_0 = 0.8619608462$, HK = 1.0300875730

B(r)	C(r)	G(r)	ψ	Fψ	90-r
1.00000 00000	1.32039 64540	0.00000 00000	90° 0′	2.03471 53122	90
0.99984 19155	1.32029 87371	0.00654 66917	89 15	2.01210 73643	89
0.99936 77261	1.32000 57060	0.01308 82806	88 31	1.98949 94164	88
0.99857 76238	1.31951 77192	0.01961 96606	87 46	1.96689 14685	87
0.99747 19280	1.31883 53734	0.02613 57182	87 1	1.94428 35205	86
0.99605 10861	1.31795 95033	0.03263 13295	86 17	1.92167 55726	85
0.99431 56720	1.31689 11801	0.03910 13564	85 32	1.89906 76247	84
0.99226 63864	1.31563 17106	0.04554 06434	84 47	1.87645 96768	83
0.98990 40553	1.31418 26349	0.05194 40144	84 2	1.85385 17289	82
0.98722 96302	1.31254 57253	0.05830 62693	83 17	1.83124 37810	81
0.98424 41861	1.31072 29838	0.06462 21812	82 32	1.80863 58331	80
0.98094 89213	1.30871 66392	0.07088 64934	81 46	1.78602 78851	79
0.97734 51558	1.30652 91449	0.07709 39167	81 1	1.76341 99372	78
0.97343 43309	1.30416 31759	0.08323 91270	80 15	1.74081 19893	77
0.96921 80039	1.30162 16250	0.08931 67629	79 29	1.71820 40414	76
0.96469 78546	1.29890 75994	0.09532 14240	78 43	1.69559 60935	75
0.95987 56758	1.29602 44173	0.10124 76688	77 56	1.67298 81456	74
0.95475 33753	1.29297 56032	0.10709 00133	77 10	1.65038 01977	73
0.94933 29736	1.28976 48840	0.11284 29301	76 23	1.62777 22497	72
0.94361 66021	1.28639 61840	0.11850 08473	75 35	1.60516 43018	71
0.93760 65006	1.28287 36204	0.12405 81487	74 48	1.58255 63539	70
0.93130 50161	1.27920 14980	0.12950 91731	74 0	1.55994 84060	69
0.92471 45098	1.27538 43041	0.13484 82153	73 12	1.53734 04581	68
0.91783 78055	1.27142 67027	0.14006 95267	72 23	1.51473 25102	67
0.91067 72870	1.26733 35291	0.14516 73172	71 35	1.49212 45623	66
0.90323 57961	1.26310 97835	0.15013 57566	70 46	1.46951 66144	65
0.89551 61707	1.25876 06253	0.15496 89777	69 56	1.44690 86665	64
0.88752 13778	1.25429 13663	0.15966 10790	69 7	1.42430 07185	63
0.87925 44206	1.24970 74646	0.16420 61290	68 16	1.40169 27706	62
0.87071 84265	1.24501 45176	0.16859 81701	67 26	1.37908 48227	61
0.86191 65988	1.24021 82552	0.17283 12244	66 35	1.35647 68748	60
0.85285 22237	1.23532 45329	0.17689 92991	65 43	1.33386 89269	59
0.84352 86672	1.23033 93242	0.18079 63935	64 51	1.31126 09790	58
0.83394 93726	1.22526 87137	0.18451 65064	63 59	1.28865 30311	57
0.82411 78578	1.22011 88895	0.18805 36444	63 6	1.26604 50832	56
0.81403 77126	1.21489 61356	0.19140 18312	62 12	1.24343 71353	55
0.80371 25960	1.20960 68240	0.19455 51177	61 19	1.22082 91873	54
0.79314 62334	1.20425 74072	0.19750 75927	60 24	1.19822 12394	53
0.78234 24136	1.19885 44102	0.20025 33955	59 30	1.17561 32915	52
0.77130 49868	1.19340 44225	0.20278 67279	58 35	1.15300 53436	51
0.76003 78612	1.18791 40899	0.20510 18688	57 39	1.13039 73957	50
0.74854 50007	1.18239 01066	0.20719 31885	56 42	1.10778 94478	49
0.73683 04220	1.17683 92068	0.20905 51650	55 46	1.08518 14999	48
0.72489 81922	1.17126 81567	0.21068 24001	54 48	1.06257 35519	47
0.71275 24260	1.16568 37461	0.21206 96370	53 50	1.03996 56041	46
0.70039 72833	1.16009 27802	0.21321 17818	52 52	1.01735 76561	45
A(r)	D(r)	E(r)	φ	Fφ	r

$K = 2.1565156475, \quad K' = 1.0867503648, \quad E = 1.211056028, \quad E' = 1.4674622093,$

r	$F\phi$	ϕ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02396 12850	1 22	0.01050 21636	1.00012 58452	0.01694 24822
2	0.04792 25699	2 45	0.02098 36904	1.00050 32288	0.03388 07351
3	0.07188 38549	4 7	0.03142 40274	1.00113 16945	0.05081 05270
4	0.09584 51399	5 29	0.04180 27880	1.00201 04822	0.06772 70275
5	0.11980 64248	6 51	0.05209 98337	1.00313 85295	0.08462 77070
6	0.14376 77098	8 13	0.06229 53533	1.00451 44723	0.10150 67944
7	0.16772 89948	9 35	0.07236 99392	1.00613 66408	0.11836 03717
8	0.19169 02798	10 56	0.08230 46606	1.00800 30911	0.13518 42734
9	0.21565 15647	12 17	0.09208 11426	1.01011 15480	0.15197 42358
10	0.23961 28497	13 38	0.10168 15801	1.01245 94672	0.16872 59855
11	0.26357 41347	14 58	0.11108 88076	1.01504 40088	0.18543 52386
12	0.28753 54197	16 18	0.12028 67034	1.01786 20403	0.20209 76999
13	0.31149 67046	17 38	0.12925 93879	1.02091 01701	0.21870 90610
14	0.33545 79896	18 57	0.13799 21503	1.02418 49023	0.23526 50037
15	0.35941 92746	20 16	0.14647 10652	1.02768 16504	0.25176 11011
16	0.38338 05595	21 35	0.15468 30530	1.03139 68120	0.26819 32750
17	0.40734 18445	22 53	0.16261 59047	1.03532 50803	0.28455 68916
18	0.43130 31295	24 10	0.17025 85702	1.03946 34091	0.30084 76617
19	0.45526 44145	25 26	0.17760 05773	1.04380 52883	0.31706 11003
20	0.47922 56994	26 42	0.18463 26382	1.04834 57003	0.33319 30668
21	0.50318 69844	27 58	0.19134 63517	1.05307 04260	0.34923 88634
22	0.52714 82694	29 13	0.19773 42593	1.05800 04010	0.36519 41381
23	0.55110 95544	30 27	0.20378 08371	1.06310 29042	0.38105 44318
24	0.57507 08393	31 41	0.20950 74827	1.06838 08291	0.39681 52701
25	0.59903 21243	32 54	0.21488 24088	1.07382 76019	0.41247 21033
26	0.62299 34093	34 7	0.21991 10718	1.07943 66784	0.42802 06069
27	0.64695 46942	35 18	0.22459 02484	1.08520 12878	0.44345 60826
28	0.67091 59792	36 29	0.22891 79082	1.09111 43480	0.45877 40585
29	0.69487 72642	37 39	0.23289 27342	1.09716 87771	0.47396 99998
30	0.71883 85492	38 49	0.23651 41807	1.10335 71080	0.48903 93230
31	0.74279 98341	39 58	0.23978 24399	1.10967 21031	0.50397 74998
32	0.76676 11191	41 6	0.24269 84060	1.11610 58243	0.51877 99184
33	0.79072 24041	42 13	0.24526 36394	1.12265 05510	0.53344 20249
34	0.81468 36890	43 20	0.24748 03283	1.12929 83450	0.54795 92224
35	0.83864 49740	44 26	0.24935 12513	1.13604 11010	0.56232 69101
36	0.86260 62590	45 31	0.25087 97387	1.14287 06563	0.57654 05212
37	0.88656 75440	46 35	0.25206 96336	1.14977 87007	0.59059 54347
38	0.91052 88289	47 39	0.25292 52540	1.15675 68364	0.60448 70073
39	0.93449 01139	48 42	0.25345 13545	1.16379 65783	0.61821 08313
40	0.95845 13989	49 44	0.25365 30884	1.17088 93642	0.63176 21451
41	0.98241 26838	50 45	0.25353 59713	1.17802 65652	0.64513 64304
42	1.00637 39688	51 46	0.25310 58450	1.18519 94950	0.65832 91446
43	1.03033 52538	52 46	0.25236 88420	1.19239 94253	0.67133 57242
44	1.05429 65388	53 45	0.25133 13558	1.19961 75873	0.68415 16433
45	1.07825 78237	54 44	0.25000 00000	1.20684 51910	0.69677 23959
90-r	$F\psi$	ψ	G(r)	C(r)	B(r)

$q = 0.0857957337021095$, $\Theta 0 = 0.8285168080$, $HK = 1.0003895588$

B(r)	C(r)	G(r)	ψ	Fψ	90-r
1.00000 00000	1.41421 35624	0.00000 00000	90° 0′	2.15651 56475	90
0.99983 87925	1.41408 70709	0.00746 45017	89 19	2.13255 43625	89
0.99935 52434	1.41370 77878	0.01492 38646	88 38	2.10859 30775	88
0.99854 95732	1.41307 61515	0.02237 29430	87 57	2.08463 17926	87
0.99742 21491	1.41219 29466	0.02980 65777	87 16	2.06067 05076	86
0.99597 34843	1.41105 92570	0.03721 95889	86 35	2.03670 92226	85
0.99420 42378	1.40967 64744	0.04460 67701	85 53	2.01274 79377	84
0.99211 54135	1.40804 62958	0.05196 28815	85 11	1.98878 66527	83
0.98970 73588	1.40617 07222	0.05928 26440	84 29	1.96482 53677	82
0.98698 17641	1.40405 20551	0.06656 07336	83 47	1.94086 40827	81
0.98393 96610	1.40169 28947	0.07379 17757	83 5	1.91690 27978	80
0.98058 24210	1.39909 61356	0.08097 03401	82 23	1.89294 15128	79
0.97691 15541	1.39626 49630	0.08809 09364	81 41	1.86898 02278	78
0.97292 87065	1.39320 28531	0.09514 80095	80 58	1.84501 89429	77
0.96863 50591	1.38991 35592	0.10213 59353	80 15	1.82105 76579	76
0.96403 43250	1.38640 11169	0.10904 90175	79 32	1.79709 63729	75
0.95912 67478	1.38266 98339	0.11588 14840	78 49	1.77313 50879	74
0.95391 50985	1.37872 42853	0.12262 74837	78 5	1.74917 38030	73
0.94840 16738	1.37456 93900	0.12928 10844	77 21	1.72521 25180	72
0.94258 88936	1.37020 99083	0.13583 62697	76 37	1.70125 12330	71
0.93647 02941	1.36565 16965	0.14228 69378	75 53	1.67728 99480	70
0.93007 55342	1.36089 99809	0.14862 68991	75 8	1.65332 86631	69
0.92338 03829	1.35596 07000	0.15484 98749	74 23	1.62936 73781	68
0.91639 67210	1.35083 98797	0.16094 94967	73 37	1.60540 60931	67
0.90912 75372	1.34554 37995	0.16691 93054	72 51	1.58144 48082	66
0.90157 50245	1.34007 89157	0.17275 27505	72 5	1.55748 35232	65
0.89374 50771	1.33445 20094	0.17844 31913	71 18	1.53352 22382	64
0.88563 82868	1.32866 98789	0.18398 38964	70 30	1.50956 09532	63
0.87725 80396	1.32273 96308	0.18936 80462	69 42	1.48559 96683	62
0.86861 05122	1.31666 85215	0.19458 87340	68 54	1.46163 83833	61
0.85969 65682	1.31046 30783	0.19963 89691	68 5	1.43767 70983	60
0.85052 07549	1.30413 35898	0.20451 16802	67 16	1.41371 58134	59
0.84108 67090	1.29768 50069	0.20919 97204	66 26	1.38975 45284	58
0.83139 85036	1.29112 63832	0.21369 58722	65 36	1.36579 32434	57
0.82145 97438	1.28446 54650	0.21799 28546	64 45	1.34183 19584	56
0.81127 44636	1.27771 04815	0.22208 33313	63 53	1.31787 06735	55
0.80084 60719	1.27086 96850	0.22595 99196	63 1	1.29390 93885	54
0.79018 04386	1.26395 14305	0.22961 52018	62 9	1.26994 81035	53
0.77927 98015	1.25696 41655	0.23304 17372	61 15	1.24598 68185	52
0.76814 92130	1.24991 64194	0.23623 20761	60 21	1.22202 55336	51
0.75679 26317	1.24281 67937	0.23917 87758	59 27	1.19806 42486	50
0.74521 44290	1.23567 39504	0.24187 44177	58 32	1.17410 29636	49
0.73341 89253	1.22849 66025	0.24431 16265	57 36	1.15014 16787	48
0.72141 04816	1.22129 35025	0.24648 30908	56 39	1.12618 03937	47
0.70919 34952	1.21407 34320	0.24838 15864	55 42	1.10221 91087	46
0.69677 23959	1.20684 51910	0.25000 00000	54 44	1.07825 78237	45
A(r)	D(r)	E(r)	φ	Fφ	r.

K = 2.3087867982, K′ = 1.6489052185, E = 1.1638279645, E′ = 1.4981149284,

r	Fφ	φ	E(r)	D(r)	Λ(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02565 31866	1 28	0.01271 71437	1.00016 31607	0.01667 62945
2	0.05130 63733	2 56	0.02540 65870	1.00065 24464	0.03334 89266
3	0.07695 95599	4 24	0.03804 07632	1.00146 72698	0.05001 42309
4	0.10261 27466	5 52	0.05059 23651	1.00260 66524	0.06666 85367
5	0.12826 59332	7 20	0.06303 44830	1.00406 92457	0.08330 81651
6	0.15391 91199	8 47	0.07534 07235	1.00585 32333	0.09992 94260
7	0.17957 23085	10 14	0.08748 53252	1.00795 65320	0.11652 86159
8	0.20522 54932	11 41	0.09944 32800	1.01037 65054	0.13310 20150
9	0.23087 86798	13 8	0.11119 04341	1.01311 05159	0.14964 58850
10	0.25653 18665	14 34	0.12270 35875	1.01615 50083	0.16615 64662
11	0.28218 50531	16 0	0.13396 05824	1.01950 64130	0.18262 99754
12	0.30783 82398	17 25	0.14494 03847	1.02316 07042	0.19906 26038
13	0.33349 14264	18 50	0.15562 31436	1.02711 34800	0.21545 05144
14	0.35914 46131	20 14	0.16599 02705	1.03136 00000	0.23178 98405
15	0.38479 77997	21 38	0.17602 44678	1.03589 31569	0.24807 66833
16	0.41045 09864	23 1	0.18570 97706	1.04071 34825	0.26430 71105
17	0.43610 41730	24 23	0.19503 16024	1.04580 91848	0.28047 71545
18	0.46175 73596	25 44	0.20397 67323	1.05117 61304	0.29658 28110
19	0.48741 05463	27 4	0.21253 33427	1.05680 78572	0.31260 00376
20	0.51306 37329	28 24	0.22069 09968	1.06269 75825	0.32858 47548
21	0.53871 69196	29 43	0.22844 06338	1.06884 82109	0.34447 28350
22	0.56437 01062	31 1	0.23577 45496	1.07522 43418	0.36028 01217
23	0.59002 32929	32 19	0.24268 63696	1.08184 22780	0.37600 24088
24	0.61567 64795	33 36	0.24917 10181	1.08869 00386	0.39163 54803
25	0.64132 96662	34 52	0.25522 46626	1.09575 73808	0.40717 40584
26	0.66698 28528	36 7	0.26084 46988	1.10303 57120	0.42261 06698
27	0.69263 60395	37 21	0.26602 96698	1.11051 63100	0.43795 68117
28	0.71828 92261	38 34	0.27077 92271	1.11819 01178	0.45318 87717
29	0.74394 24127	39 46	0.27509 40704	1.12604 78614	0.46831 04485
30	0.76959 55994	40 58	0.27897 58872	1.13408 09433	0.48331 61880
31	0.79524 87860	42 9	0.28242 72020	1.14227 69496	0.49820 24170
32	0.82090 19727	43 18	0.28545 17629	1.15062 86634	0.51296 30449
33	0.84655 51593	44 26	0.28808 35786	1.15912 80752	0.52759 55047
34	0.87220 83460	45 34	0.29023 77551	1.16775 89904	0.54208 35352
35	0.89786 15326	46 41	0.29200 99840	1.17651 06708	0.55645 28824
36	0.92351 47193	47 47	0.29337 65050	1.18537 87860	0.57066 80018
37	0.94916 79059	48 52	0.29434 43597	1.19434 91887	0.58473 68014
38	0.97482 10926	49 56	0.29492 07141	1.20341 18951	0.59868 20033
39	1.00047 42792	50 59	0.29511 34150	1.21255 50050	0.61240 95465
40	1.02612 74659	52 1	0.29493 06347	1.22176 77148	0.62600 46907
41	1.05178 06525	53 2	0.29438 08705	1.23103 88308	0.63943 20185
42	1.07743 38392	54 2	0.29347 29047	1.24035 70830	0.65268 84902
43	1.10308 70258	55 1	0.29221 57532	1.24971 11383	0.66576 74022
44	1.12874 02125	56 0	0.29061 86227	1.25908 96145	0.67866 47507
45	1.15439 33991	56 58	0.28869 08691	1.26848 10938	0.69137 54254
90−r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.106054020186904$, $\Theta 0 = 0.7881449667$, $HK = 1.1541701350$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	1.53824 62687	0.00000 00000	90° 0′	2.30878 67982	90
0.99983 41412	1.53808 15440	0.00834 87781	88 23	2.28313 36115	89
0.99933 66526	1.53758 75740	0.01669 26008	88 46	2.25748 04249	88
0.99850 77970	1.53676 49688	0.02502 65041	88 9	2.23182 72382	87
0.99734 80125	1.53561 47447	0.03334 55075	87 32	2.20617 40516	86
0.99585 79109	1.53413 83232	0.04164 46052	86 54	2.18052 08649	85
0.99403 82778	1.53233 75281	0.04991 87582	86 16	2.15486 76783	84
0.99189 00707	1.53021 45843	0.05816 28855	85 38	2.12921 44916	83
0.98941 44182	1.52777 21140	0.06637 18564	85 0	2.10356 13050	82
0.98661 26176	1.52501 31340	0.07454 04819	84 22	2.07790 81184	81
0.98348 61339	1.52194 10514	0.08266 35068	83 44	2.05225 49317	80
0.98003 65979	1.51855 96596	0.09073 56016	83 6	2.02660 17451	79
0.97626 57996	1.51487 31329	0.09875 13547	82 27	2.00094 85584	78
0.97217 56947	1.51088 60218	0.10670 52642	81 48	1.97529 53718	77
0.96776 83924	1.50660 32466	0.11459 17308	81 9	1.94964 21851	76
0.96304 61576	1.50203 09916	0.12240 50500	80 30	1.92398 89985	75
0.95801 14969	1.49717 21977	0.13013 94047	79 50	1.89833 58118	74
0.95266 67013	1.49203 55559	0.13778 88583	79 10	1.87268 26251	73
0.94701 47511	1.48662 64993	0.14534 73477	78 30	1.84702 94385	72
0.94105 84035	1.48095 10947	0.15280 86769	77 49	1.82137 62519	71
0.93480 06429	1.47501 81348	0.16016 65105	77 8	1.79572 30652	70
0.92824 45859	1.46883 31288	0.16741 43683	76 26	1.77006 98786	69
0.92139 34772	1.46240 42933	0.17454 56190	75 44	1.74441 66919	68
0.91425 06851	1.45573 95424	0.18155 34763	75 2	1.71876 35053	67
0.90681 96968	1.44884 70781	0.18843 09933	74 19	1.69311 03186	66
0.89910 41140	1.44173 53793	0.19517 10594	73 36	1.66745 71320	65
0.89110 76479	1.43441 31916	0.20176 63966	72 52	1.64180 39453	64
0.88283 41144	1.42688 95162	0.20820 95570	72 8	1.61615 07587	63
0.87428 74294	1.41917 35981	0.21449 29211	71 23	1.59049 75721	62
0.86547 16034	1.41127 49149	0.22060 86968	70 37	1.56484 43854	61
0.85639 07366	1.40320 31647	0.22654 89197	69 51	1.53919 11988	60
0.84704 90138	1.39496 82541	0.23230 54536	69 4	1.51353 80121	59
0.83745 06991	1.38658 02852	0.23786 99932	68 17	1.48788 48255	58
0.82760 01310	1.37804 95440	0.24323 40676	67 29	1.46223 16388	57
0.81750 17168	1.36938 64865	0.24838 90447	66 41	1.43657 84522	56
0.80715 99276	1.36060 17261	0.25332 61379	65 52	1.41092 52655	55
0.79657 92934	1.35170 60205	0.25803 64133	65 2	1.38527 20789	54
0.78576 43973	1.34271 02582	0.26251 08001	64 11	1.35961 88922	53
0.77471 98768	1.33362 54449	0.26674 01012	63 20	1.33396 57055	52
0.76345 03880	1.32446 26960	0.27071 50065	62 28	1.30831 25189	51
0.75196 06646	1.31523 31927	0.27442 61086	61 35	1.28265 93322	50
0.74025 54413	1.30594 82284	0.27786 39198	60 41	1.25700 61456	49
0.72833 95927	1.29661 91348	0.28101 88920	59 46	1.23135 29589	48
0.71621 76383	1.28725 72976	0.28388 14388	58 51	1.20569 97723	47
0.70389 46686	1.27787 41372	0.28644 19600	57 55	1.18004 65856	46
0.69137 54254	1.26848 10938	0.28869 08691	56 58	1.15439 33991	45
A(r)	D(r)	E(r)	φ	Fφ	r

K = 2.5045500790, K′ = 1.6200258991, E = 1.1183777380, E′ = 1.5237002053,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.02782 83342	1 36	0.01539 55735	1.00021 42837	0.01627 42346
2	0.05565 66684	3 11	0.03075 31420	1.00085 68806	0.03254 56619
3	0.08348 50026	4 47	0.04603 49252	1.00192 70204	0.04881 14608
4	0.11131 33368	6 22	0.06120 35769	1.00342 34614	0.06506 88358
5	0.13914 16710	7 57	0.07622 24069	1.00534 44038	0.08131 49227
6	0.16697 00053	9 32	0.09105 55815	1.00768 75763	0.09754 68734
7	0.19479 83395	11 6	0.10566 83103	1.01045 02042	0.11376 18057
8	0.22262 66737	12 40	0.12002 70732	1.01362 90072	0.12995 68083
9	0.25045 50079	14 13	0.13409 96084	1.01722 02172	0.14612 80355
10	0.27828 33421	15 46	0.14785 56040	1.02121 95717	0.16227 52039
11	0.30611 16763	17 18	0.16120 58874	1.02562 23337	0.17840 25828
12	0.33394 00105	18 50	0.17430 34501	1.03042 32454	0.19447 80006
13	0.36176 83447	20 20	0.18694 30048	1.03561 66441	0.21052 83497
14	0.38959 66790	21 50	0.19916 16028	1.04119 63185	0.22654 03885
15	0.41742 50132	23 20	0.21093 77018	1.04715 56657	0.24251 00363
16	0.44525 33474	24 48	0.22225 25549	1.05348 75877	0.25843 66097
17	0.47308 16816	26 16	0.23308 88866	1.06018 45500	0.27431 44196
18	0.50091 00158	27 42	0.24343 18557	1.06723 85705	0.29014 01480
19	0.52873 83500	29 8	0.25326 86498	1.07464 13734	0.30591 09453
20	0.55656 66842	30 32	0.26258 84863	1.08238 48086	0.32162 30277
21	0.58439 50184	31 56	0.27138 25008	1.09045 69513	0.33727 27340
22	0.61222 33526	33 18	0.27964 41053	1.09885 10674	0.35285 63285
23	0.64005 16869	34 40	0.28736 82581	1.10755 61340	0.36836 99898
24	0.66788 00211	36 0	0.29455 17462	1.11656 17464	0.38380 98186
25	0.69570 83553	37 19	0.30110 32185	1.12585 71488	0.39917 18323
26	0.72353 66895	38 37	0.30720 28884	1.13544 11869	0.41445 19649
27	0.75136 50237	39 54	0.31285 24053	1.14527 24456	0.42964 60668
28	0.77919 33579	41 10	0.31787 52022	1.15536 09697	0.44474 90013
29	0.80702 16921	42 24	0.32236 54011	1.16570 89825	0.45975 91601
30	0.83485 00263	43 38	0.32632 90569	1.17627 97795	0.47466 94339
31	0.86267 83605	44 50	0.32977 27014	1.18706 87529	0.48947 62428
32	0.89050 66948	46 1	0.33270 42283	1.19806 29307	0.50417 80229
33	0.91833 50290	47 11	0.33513 23398	1.20924 90840	0.51876 11309
34	0.94616 33632	48 20	0.33706 05364	1.22061 37375	0.53322 98456
35	0.97399 16974	49 27	0.33851 70194	1.23214 31046	0.54757 63701
36	1.00182 00316	50 34	0.33949 45075	1.24382 35438	0.56179 88348
37	1.02964 83658	51 39	0.34001 05978	1.25564 06708	0.57588 32096
38	1.05747 67000	52 43	0.34007 67814	1.26758 03104	0.58983 37576
39	1.08530 50342	53 46	0.33970 52640	1.27962 80178	0.60364 21381
40	1.11313 33684	54 48	0.33890 84414	1.29176 01861	0.61730 33109
41	1.14096 17027	55 49	0.33769 80203	1.30398 91085	0.63081 20807
42	1.16879 00369	56 48	0.33608 94543	1.31627 20590	0.64416 32373
43	1.19661 83711	57 47	0.33409 28851	1.32860 58237	0.65735 14695
44	1.22444 67053	58 44	0.33172 20892	1.34097 27096	0.67037 14605
45	1.25227 50395	59 41	0.32898 09283	1.35335 85717	0.68321 78479
90-r	Fψ	ψ	G(r)	C(r)	B(r)

q = 0.131061824400858, () 0 = 0.7384064407, HK = 1.2240462555

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	1.70991 35051	0.00000 00000	90° 0′	2.50455 00790	90
0.99982 71058	1.70969 53883	0.00917 03805	89 27	2.47672 17448	89
0.99930 85325	1.70904 11308	0.01833 63062	88 55	2.44889 34106	88
0.99844 40074	1.70795 16110	0.02749 33119	88 22	2.42106 50764	87
0.99723 58755	1.70642 81917	0.03663 69110	87 49	2.39323 67422	86
0.99568 30084	1.70447 27784	0.04576 25853	87 16	2.36540 84079	85
0.99378 72533	1.70208 78163	0.05486 57745	86 43	2.33758 00737	84
0.99154 95309	1.69927 62875	0.06394 18650	86 10	2.30975 17395	83
0.98897 13334	1.69604 17067	0.07298 61798	85 36	2.28192 34053	82
0.98605 42725	1.69238 81168	0.08199 39678	85 3	2.25409 50711	81
0.98280 01661	1.68832 00831	0.09096 03928	84 29	2.22626 67369	80
0.97921 10350	1.68384 26872	0.09988 05231	83 55	2.19843 84027	79
0.97528 91023	1.67896 15207	0.10874 93206	83 21	2.17061 00685	78
0.97103 67835	1.67368 26771	0.11756 16303	82 46	2.14278 17343	77
0.96645 66885	1.66801 27439	0.12631 21691	82 12	2.11495 34000	76
0.96155 10144	1.66195 87040	0.13499 55158	81 37	2.08712 50658	75
0.95632 45409	1.65552 83761	0.14360 66995	81 1	2.05929 67316	74
0.95077 86259	1.64872 95046	0.15213 81898	80 25	2.03146 83974	73
0.94491 71996	1.64157 06491	0.16058 58855	79 49	2.00364 00632	72
0.93874 37597	1.63406 07230	0.16894 31044	79 13	1.97581 17290	71
0.93226 19647	1.62620 90730	0.17720 35729	78 36	1.94798 33948	70
0.92547 56289	1.61802 54015	0.18536 08158	77 58	1.92015 50606	69
0.91838 87155	1.60952 00037	0.19340 81461	77 20	1.89232 67264	68
0.91100 53304	1.60070 34445	0.20133 86551	76 42	1.86449 83921	67
0.90332 97156	1.59158 65494	0.20914 52034	76 3	1.83667 00579	66
0.89536 62423	1.58218 06891	0.21682 04110	75 23	1.80884 17237	65
0.88711 94043	1.57249 75252	0.22435 66494	74 43	1.78101 33895	64
0.87859 38106	1.56254 90544	0.23174 60328	74 2	1.75318 50553	63
0.86979 41783	1.55234 75933	0.23898 04111	73 21	1.72535 67211	62
0.86072 53257	1.54190 57623	0.24605 13624	72 39	1.69752 83869	61
0.85139 21644	1.53123 64694	0.25295 01875	71 56	1.66970 00527	60
0.84179 96923	1.52035 28933	0.25966 79043	71 13	1.64187 17185	59
0.83195 29861	1.50926 84668	0.26619 52443	70 29	1.61404 33842	58
0.82185 71038	1.49799 68598	0.27252 26492	69 44	1.58621 50500	57
0.81151 75269	1.48655 19691	0.27864 02697	68 59	1.55838 67158	56
0.80093 92537	1.47494 78592	0.28453 79654	68 12	1.53055 83816	55
0.79012 76914	1.46319 88308	0.29020 53069	67 25	1.50273 00474	54
0.77908 81986	1.45131 93148	0.29563 15786	66 37	1.47490 17132	53
0.76782 61083	1.43932 38985	0.30080 57852	65 48	1.44707 33790	52
0.75634 70207	1.42722 72983	0.30571 66593	64 59	1.41924 50448	51
0.74465 61957	1.41504 43413	0.31035 26720	64 8	1.39141 67106	50
0.73275 91466	1.40278 99470	0.31470 20462	63 17	1.36358 83763	49
0.72066 13327	1.39047 91083	0.31875 27727	62 24	1.33576 00421	48
0.70836 82126	1.37812 68735	0.32249 26298	61 31	1.30793 17079	47
0.69588 52382	1.36574 83271	0.32590 92064	60 36	1.28010 33737	46
0.68321 78479	1.35335 85717	0.32898 99283	59 41	1.25227 50395	45
A(r)	D(r)	E(r)	φ	Fφ	r

$K = 2.7680631454 = K'\sqrt{3}$, $K' = 1.5981420021$, $E = 1.076405113$, $E' = 1.5441504009$,

r	$F\phi$	ϕ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.03075 62572	1 46	0.01878 71553	1.00028 90226	0.01564 67728
2	0.06151 25143	3 37	0.03752 01201	1.00115 57568	0.03129 20711
3	0.09226 87715	5 17	0.05614 59985	1.00259 92025	0.04693 44040
4	0.12302 50287	7 2	0.07460 90790	1.00461 76035	0.06257 22754
5	0.15378 12859	8 47	0.09286 02100	1.00720 88007	0.07820 41558
6	0.18453 75430	10 31	0.11084 81632	1.01036 98288	0.09382 84843
7	0.21529 38002	12 15	0.12852 44620	1.01409 68205	0.10944 30574
8	0.24605 00574	13 58	0.14584 27086	1.01838 55046	0.12504 80220
9	0.27680 63145	15 40	0.16275 93073	1.02323 11658	0.14063 98605
10	0.30756 25717	17 22	0.17923 28003	1.02862 70374	0.15621 74137
11	0.33831 88289	19 3	0.19522 50184	1.03456 96626	0.17177 88130
12	0.36907 50860	20 43	0.21070 07095	1.04104 94593	0.18732 21327
13	0.39983 13432	22 22	0.22562 78479	1.04805 98103	0.20284 53538
14	0.43058 76004	23 59	0.23997 76707	1.05559 26010	0.21834 63622
15	0.46134 38576	25 36	0.25372 47838	1.06363 90673	0.23382 29130
16	0.49210 01147	27 12	0.26684 70884	1.07218 98612	0.24927 27730
17	0.52285 63719	28 46	0.27932 58519	1.08123 50440	0.26469 34104
18	0.55361 26291	30 19	0.29114 56129	1.09076 40755	0.28008 23255
19	0.58436 88862	31 50	0.30229 41110	1.10076 58484	0.29543 68145
20	0.61512 51434	33 21	0.31276 21816	1.11122 86903	0.31075 40803
21	0.64588 14006	34 50	0.32254 36297	1.12214 04756	0.32603 11842
22	0.67663 76577	36 17	0.33163 50828	1.13348 81362	0.34126 50509
23	0.70739 39149	37 43	0.34003 58309	1.14525 86847	0.35645 24053
24	0.73815 01721	39 8	0.34774 76532	1.15743 82078	0.37159 00604
25	0.76890 64293	40 31	0.35477 46364	1.17001 24008	0.38667 43509
26	0.79966 26864	41 52	0.36112 29881	1.18296 64722	0.40170 16862
27	0.83041 89436	43 12	0.36680 08467	1.19628 51612	0.41666 82489
28	0.86117 52008	44 31	0.37181 80918	1.20995 27538	0.43157 00088
29	0.89193 14579	45 48	0.37618 01563	1.22395 30095	0.44640 31361
30	0.92268 77151	47 3	0.37991 78428	1.23826 96288	0.46116 31110
31	0.95344 39723	48 18	0.38302 71460	1.25288 83092	0.47584 56238
32	0.98420 02294	49 30	0.38552 90817	1.26778 20672	0.49044 01259
33	1.01495 64866	50 41	0.38743 95246	1.28294 47038	0.50495 09214
34	1.04571 27438	51 51	0.38877 50552	1.29835 28184	0.51938 21608
35	1.07646 90010	52 59	0.38955 28159	1.31398 80140	0.53370 78866
36	1.10722 52581	54 5	0.38979 03785	1.32983 25072	0.54793 19494
37	1.13798 15153	55 10	0.38950 56204	1.34586 70108	0.56204 99089
38	1.16873 77725	56 14	0.38871 66125	1.36207 23140	0.57605 39442
39	1.19949 40296	57 16	0.38744 15171	1.37842 80138	0.58994 19669
40	1.23025 02868	58 17	0.38569 84955	1.39491 71251	0.60370 45207
41	1.26100 65440	59 17	0.38350 56260	1.41151 70596	0.61733 88663
42	1.29176 28011	60 15	0.38088 08305	1.42820 86570	0.63084 81179
43	1.32251 90583	61 12	0.37784 18107	1.44497 17132	0.64419 63062
44	1.35327 53155	62 8	0.37440 59023	1.46178 58952	0.65740 73705
45	1.38403 15727	63 2	0.37059 04774	1.47863 07741	0.67046 51423
90-r	$F\psi$	ψ	G(r)	C(r)	B(r)

TABLE $\theta = 75°$

$q = 0.163033534821580$, $\Theta 0 = 0.0753457533$, $HK = 1.3046678096$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	1.96563 05108	0.00000 00000	90° 0′	2.76806 31454	90
0.99981 60886	1.96533 12951	0.00989 91720	89 33	2.73730 68882	89
0.99926 44975	1.96443 40309	0.01979 47043	89 5	2.70655 06310	88
0.99834 56552	1.96293 98674	0.02968 29453	88 38	2.67579 43738	87
0.99706 02753	1.96085 07176	0.03956 02195	88 10	2.64503 81167	86
0.99540 93546	1.95816 92561	0.04942 28154	87 43	2.61428 18595	85
0.99339 41714	1.95489 89147	0.05926 69738	87 15	2.58352 56023	84
0.99101 62829	1.95104 38778	0.06908 88752	86 47	2.55276 93451	83
0.98827 75221	1.94660 90763	0.07888 46278	86 19	2.52201 30880	82
0.98517 99040	1.94160 01803	0.08865 02550	85 51	2.49125 68308	81
0.98172 60720	1.93602 35909	0.09838 16828	85 22	2.46050 05736	80
0.97791 83023	1.92988 64309	0.10807 47268	84 54	2.42974 43165	79
0.97375 98498	1.92319 65349	0.11772 50798	84 25	2.39898 80593	78
0.96925 35914	1.91596 24373	0.12732 82981	83 55	2.36823 18021	77
0.96440 30166	1.90819 33608	0.13687 97883	83 26	2.33747 55450	76
0.95921 17495	1.89989 92030	0.14637 47936	82 56	2.30671 92878	75
0.95368 36468	1.89109 05214	0.15580 83802	82 25	2.27596 30306	74
0.94782 28900	1.88177 85195	0.16517 54225	81 55	2.24520 67734	73
0.94163 35686	1.87197 50301	0.17447 05894	81 24	2.21445 05163	72
0.93512 04092	1.86169 24091	0.18368 83293	80 52	2.18369 42591	71
0.92828 80593	1.85094 39670	0.19282 28550	80 20	2.15293 80019	70
0.92114 14474	1.83974 30516	0.20186 81293	79 48	2.12218 17448	69
0.91368 56040	1.82810 39279	0.21081 78488	79 15	2.09142 54876	68
0.90592 58521	1.81604 13089	0.21966 54291	78 41	2.06066 92304	67
0.89786 75972	1.80357 04247	0.22840 39887	78 7	2.02991 29733	66
0.88951 64174	1.79070 70015	0.23702 63334	77 32	1.99915 67161	65
0.88087 80328	1.77746 72401	0.24552 49406	76 56	1.96840 04589	64
0.87195 82052	1.76386 77929	0.25389 19433	76 20	1.93764 42017	63
0.86276 31773	1.74992 57419	0.26211 91147	75 43	1.90688 79446	62
0.85329 87622	1.73565 88746	0.27019 78524	75 6	1.87613 16874	61
0.84357 12322	1.72108 41609	0.27811 91636	74 27	1.84537 54302	60
0.83358 65580	1.70622 07286	0.28587 36500	73 48	1.81461 91731	59
0.82335 19876	1.69108 68389	0.29345 14936	73 8	1.78386 29159	58
0.81287 30383	1.67570 13618	0.30084 24433	72 28	1.75310 66587	57
0.80215 64710	1.66008 34507	0.30803 58026	71 46	1.72235 04016	56
0.79120 88085	1.64425 25175	0.31502 04176	71 4	1.69159 41444	55
0.78003 68055	1.62822 82065	0.32178 46673	70 20	1.66083 78872	54
0.76864 64021	1.61203 03692	0.32831 64547	69 36	1.63008 16300	53
0.75704 48103	1.59567 90385	0.33460 32006	68 50	1.59932 53729	52
0.74523 84036	1.57919 44025	0.34063 18384	68 4	1.56856 91157	51
0.73323 37566	1.56259 67789	0.34638 88130	67 16	1.53781 28585	50
0.72103 74248	1.54590 65890	0.35186 00808	66 28	1.50705 66014	49
0.70865 59347	1.52914 43320	0.35703 11148	65 38	1.47630 03442	48
0.69609 57739	1.51233 05588	0.36188 69115	64 47	1.44554 40870	47
0.68336 33823	1.49548 58469	0.36641 20039	63 55	1.41478 78299	46
0.67046 81423	1.47863 07744	0.37059 04774	63 2	1.38403 15727	45
A(r)	D(r)	E(r)	φ	Fφ	r

K = 3.1533852519, K′ = 1.5828428043, E = 1.0401143957, E′ · 1.5588871966,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.03503 76139	2	0	0.02346 68886	1.00041 13182	0.01460 06854
2	0.07007 52278	4	1	0.04685 05457	1.00164 48264	0.02920 20956
3	0.10511 28417	6	1	0.07006 85417	1.00369 91860	0.04380 49412
4	0.14015 04556	8	0	0.09304 00333	1.00657 21668	0.05840 99043
5	0.17518 80695	9	59	0.11568 65173	1.01026 06485	0.07301 76251
6	0.21022 56835	11	58	0.13793 25365	1.01476 06225	0.08762 86871
7	0.24526 32974	13	55	0.15970 63263	1.02006 71948	0.10224 36040
8	0.28030 09113	15	52	0.18094 03901	1.02617 48886	0.11686 28061
9	0.31533 85252	17	47	0.20157 19949	1.03307 61484	0.13148 66263
10	0.35037 61391	19	41	0.22154 35813	1.04076 43440	0.14611 52882
11	0.38541 37530	21	34	0.24080 30831	1.04923 07759	0.16074 88022
12	0.42045 13669	23	26	0.25930 41559	1.05846 61800	0.17538 74010
13	0.45548 89808	25	16	0.27700 63163	1.06846 04345	0.19003 06422
14	0.49052 65947	27	4	0.29387 49943	1.07920 25667	0.20467 82669
15	0.52556 42086	28	51	0.30988 15035	1.09068 07598	0.21932 97086
16	0.56060 18226	30	36	0.32500 29380	1.10288 23022	0.23398 44577
17	0.59563 94365	32	20	0.33922 20017	1.11579 38955	0.24864 14540
18	0.63067 70504	34	1	0.35252 67798	1.12940 10647	0.26329 96770
19	0.66571 46643	35	41	0.36491 04618	1.14368 87684	0.27795 78408
20	0.70075 22782	37	18	0.37637 10249	1.15864 11101	0.29261 44375
21	0.73578 98921	38	54	0.38691 08879	1.17424 14105	0.30726 77376
22	0.77082 75060	40	28	0.39653 65430	1.19047 22106	0.32191 57797
23	0.80586 51199	41	59	0.40525 81757	1.20731 53312	0.33655 63648
24	0.84090 27338	43	29	0.41308 92784	1.22475 17970	0.35118 70467
25	0.87594 03477	44	56	0.42004 62655	1.24276 10431	0.36580 51367
26	0.91097 79617	46	22	0.42614 80665	1.26134 83814	0.38040 76896
27	0.94601 55756	47	45	0.43141 59095	1.28042 10369	0.39499 15950
28	0.98105 31895	49	7	0.43587 26721	1.30002 71887	0.40955 31244
29	1.01609 08034	50	26	0.43954 28505	1.32012 14204	0.42408 88287
30	1.05112 84173	51	44	0.44245 21005	1.34068 08140	0.43859 40475
31	1.08616 60312	52	59	0.44462 69813	1.36168 10608	0.45306 63090
32	1.12120 36451	54	12	0.44609 46031	1.38309 86803	0.46749 93408
33	1.15624 12590	55	24	0.44688 28394	1.40490 86089	0.48188 86609
34	1.19127 88729	56	33	0.44701 92128	1.42708 84443	0.49623 01775
35	1.22631 64868	57	41	0.44653 16053	1.44960 33004	0.51051 76900
36	1.26135 41008	58	47	0.44544 76404	1.47243 88241	0.52474 80842
37	1.29639 17147	59	51	0.44379 46284	1.49555 61410	0.53890 62878
38	1.33142 93286	60	53	0.44159 94403	1.51893 69731	0.55300 15038
39	1.36646 69425	61	54	0.43888 84024	1.54255 06233	0.56701 60378
40	1.40150 45564	62	53	0.43568 72080	1.56636 90138	0.58094 80681
41	1.43654 21703	63	50	0.43202 08450	1.59036 37173	0.59478 89867
42	1.47157 97842	64	45	0.42791 35381	1.61450 59885	0.60853 26019
43	1.50661 73981	65	39	0.42338 87053	1.63876 67967	0.62217 18423
44	1.54165 50120	66	32	0.41846 89243	1.66311 68595	0.63569 93846
45	1.57669 26259	67	23	0.41317 59112	1.68752 66770	0.64910 77848
90−r	Fψ	ψ		G(r)	C(r)	B(r)

$q = 0.206609755200906$, ΘO = 0.500423678356, HK = 1.400061408420

B(r)	C(r)	G(r)	ψ	Fψ	90–r
1.00000 00000	2.30974 48470	0.00000 00000	90° 0′	3.15338 52519	90
0.99979 75519	2.30940 24161	0.01049 98030	89 39	3.11834 76380	89
0.99919 04200	2.30797 88675	0.02099 72001	89 18	3.08331 00241	88
0.99817 91061	2.30577 48778	0.03148 95052	88 57	3.04827 24102	87
0.99676 48843	2.30260 34301	0.04197 43187	88 36	3.01323 47963	86
0.99494 88778	2.29873 86703	0.05244 88508	88 15	2.97819 71823	85
0.99273 29703	2.29391 59122	0.06291 05550	87 54	2.94315 95684	84
0.99011 03406	2.28823 16619	0.07335 67304	87 32	2.90812 19545	83
0.98711 05531	2.28169 33954	0.08378 46353	87 11	2.87308 43406	82
0.98370 05524	2.27430 09572	0.09419 13935	86 49	2.83804 67267	81
0.97991 96536	2.26609 12580	0.10457 40674	86 27	2.80300 91128	80
0.97574 45480	2.25704 82441	0.11492 96001	86 4	2.76797 14989	79
0.97118 82444	2.24719 29913	0.12525 49140	85 42	2.73293 38850	78
0.96625 51552	2.23653 86504	0.13554 63814	85 19	2.69789 62711	77
0.96094 99073	2.22509 91002	0.14580 08404	84 56	2.66285 86572	76
0.95527 78900	2.21289 01563	0.15601 45490	84 32	2.62782 10432	75
0.94924 39913	2.19992 86408	0.16618 36848	84 8	2.59278 34293	74
0.94285 41842	2.18622 96097	0.17630 44256	83 44	2.55774 58154	73
0.93611 43595	2.17151 24201	0.18637 19320	83 19	2.52270 82015	72
0.92904 07633	2.15601 64112	0.19638 23208	82 54	2.48767 05876	71
0.92160 99031	2.14000 12139	0.20633 09015	82 28	2.45263 29737	70
0.91385 85465	2.12411 76139	0.21621 30167	82 1	2.41759 53598	69
0.90578 30091	2.10735 72194	0.22602 10124	81 35	2.38255 77459	68
0.89739 25015	2.07095 21214	0.23575 20713	81 7	2.34752 01320	67
0.88869 44749	2.04135 63602	0.24539 02508	80 39	2.31248 25181	66
0.87969 11946	2.01249 20248	0.25495 62404	80 10	2.27744 49041	65
0.87039 61511	1.98308 66206	0.26441 63838	79 41	2.24240 72902	64
0.86081 64996	1.96316 26690	0.27377 25636	79 11	2.20736 96763	63
0.85095 85006	1.95374 16407	0.28301 72073	78 40	2.17233 20624	62
0.84084 15928	1.96156 84052	0.29214 25142	78 8	2.13729 44485	61
0.83044 47863	2.01054 51606	0.30114 98388	77 35	2.10225 68346	60
0.81980 41906	2.01881 41740	0.31000 92630	77 2	2.06721 92207	59
0.80891 61586	1.99000 04165	0.31871 42670	76 28	2.03218 16068	58
0.79779 91091	1.97132 94025	0.32727 17611	75 52	1.99714 39929	57
0.78645 27012	1.95113 27275	0.33566 20564	75 16	1.96210 63790	56
0.77488 73049	1.92844 86823	0.34387 38337	74 39	1.92706 87650	55
0.76311 41867	1.90462 82611	0.35189 81171	74 1	1.89203 11511	54
0.75113 24717	1.88147 30026	0.35971 32414	73 21	1.85699 35372	53
0.73896 02579	1.85730 11210	0.36731 48250	72 41	1.82195 59233	52
0.72661 79867	1.83187 60425	0.37468 57413	71 59	1.78691 83094	51
0.71408 92521	1.80612 88403	0.38181 10919	71 16	1.75188 06955	50
0.70139 44563	1.78516 78703	0.38867 81812	70 32	1.71684 30816	49
0.68854 06238	1.76081 71390	0.39526 14938	69 47	1.68180 54677	48
0.67554 62478	1.73640 67603	0.40155 26735	69 0	1.64676 78538	47
0.66238 94608	1.71106 68068	0.40753 05071	68 12	1.61173 02399	46
0.64910 77848	1.68742 16770	0.41317 59112	67 23	1.57669 26259	45
[illegible]	[illegible]	[illegible]	φ	Fφ	r

$K = 3.2553029421$, $K' = 1.5805400339$, $E = 1.033789462$, $E' = 1.5611417453$,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.03617 00327	2 4	0.02400 81037	1.00044 63017	0.01430 61216
2	0.07234 00654	4 8	0.04924 41210	1.00178 40728	0.02861 35824
3	0.10851 00981	6 12	0.07303 60132	1.00401 41114	0.04292 37056
4	0.14468 01308	8 16	0.09775 72158	1.00713 23080	0.05723 77835
5	0.18085 01635	10 18	0.12151 85252	1.01113 53504	0.07155 70600
6	0.21702 01961	12 20	0.14483 70258	1.01601 02772	0.08588 27206
7	0.25319 02288	14 21	0.16763 68426	1.02177 88885	0.10021 58677
8	0.28936 02615	16 21	0.18984 17049	1.02840 80140	0.11455 75144
9	0.32553 02942	18 20	0.21138 45101	1.03589 96677	0.12890 85656
10	0.36170 03269	20 18	0.23220 32821	1.04424 57511	0.14326 98042
11	0.39787 03596	22 14	0.25224 24183	1.05343 73577	0.15764 18707
12	0.43404 03923	24 8	0.27145 29257	1.06346 40252	0.17202 52803
13	0.47021 04250	26 1	0.28979 25185	1.07431 07854	0.18642 03484
14	0.50638 04577	27 53	0.30722 57913	1.08598 21410	0.20082 72302
15	0.54255 04904	29 42	0.32372 38467	1.09844 81017	0.21524 59210
16	0.57872 05230	31 29	0.33926 44387	1.11170 11775	0.22967 61638
17	0.61489 05557	33 15	0.35383 15791	1.12572 06891	0.24411 75248
18	0.65106 05884	34 58	0.36741 82531	1.14051 02773	0.25856 03307
19	0.68723 06211	36 40	0.38001 11223	1.15603 10127	0.27303 07120
20	0.72340 06538	38 19	0.39162 00536	1.17228 30068	0.28750 05037
21	0.75957 06865	39 56	0.40224 77358	1.18923 01126	0.30197 73260
22	0.79574 07192	41 32	0.41190 42230	1.20688 27770	0.31648 05358
23	0.83191 07519	43 4	0.42060 31838	1.22519 44855	0.33094 84105
24	0.86808 07846	44 35	0.42836 29363	1.24415 42355	0.34544 21058
25	0.90425 08173	46 4	0.43520 30077	1.26374 00274	0.35991 80053
26	0.94042 08500	47 30	0.44114 66047	1.28393 20625	0.37440 99070
27	0.97659 08826	48 54	0.44621 91466	1.30470 66611	0.38887 48743
28	1.01276 09153	50 16	0.45044 72717	1.32604 00103	0.40333 05918
29	1.04893 09480	51 36	0.45385 93083	1.34790 83334	0.41779 01532
30	1.08510 09807	52 54	0.45648 47848	1.37028 65097	0.43222 35590
31	1.12127 10134	54 9	0.45835 36084	1.39314 94160	0.44663 47290
32	1.15744 10461	55 23	0.45949 63841	1.41647 02902	0.46101 04825
33	1.19361 10788	56 34	0.45994 38581	1.44022 48696	0.47537 20905
34	1.22978 11115	57 43	0.45972 67648	1.46438 12257	0.48969 02410
35	1.26595 11442	58 51	0.45887 56209	1.48891 48802	0.50396 58883
36	1.30212 11769	59 56	0.45742 05610	1.51379 62870	0.51819 42606
37	1.33829 12095	61 0	0.45539 11968	1.53899 63603	0.53236 05303
38	1.37446 12422	62 2	0.45281 64872	1.56448 55101	0.54648 54602
39	1.41063 12749	63 1	0.44972 46168	1.59024 37776	0.56054 36107
40	1.44680 13076	64 0	0.44614 30615	1.61621 09676	0.57451 42929
41	1.48297 13403	64 56	0.44209 82256	1.64238 50248	0.58841 15607
42	1.51914 13730	65 51	0.43761 56044	1.66872 55844	0.60222 32286
43	1.55531 14057	66 44	0.43272 06504	1.69520 15399	0.61594 08828
44	1.59148 14384	67 35	0.42743 48807	1.72178 06994	0.62955 65890
45	1.62765 14711	68 25	0.42178 27678	1.74842 93662	0.64306 34108
90-r	Fψ	ψ	C(r)		

$q = 0.217548940000726$, $OO = 0.5093707108$, $IIK = 1.4306906219$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	2.52843 01251	0.00000 00000	90° 0′	3.25530 29421	90
0.99970 22836	2.52784 54320	0.01060 10292	89 41	3.21913 29095	89
0.99916 93515	2.52649 20136	0.02119 07963	89 21	3.18296 28768	88
0.99813 18540	2.52407 18500	0.03179 40278	89 2	3.14679 28441	87
0.99668 08744	2.52058 82420	0.04238 14278	88 42	3.11062 28114	86
0.99481 70213	2.51624 57960	0.05295 96662	88 22	3.07445 27787	85
0.99254 49353	2.51095 01254	0.06352 63677	88 2	3.03828 27460	84
0.98986 42745	2.50470 03354	0.07407 99993	87 42	3.00211 27133	83
0.98677 87139	2.49753 10120	0.08461 53590	87 22	2.96594 26806	82
0.98329 14083	2.48942 82067	0.09513 25631	87 2	2.92977 26479	81
0.97940 60341	2.48040 20203	0.10562 80337	86 41	2.89360 26152	80
0.97512 69836	2.47047 03835	0.11609 89854	86 20	2.85743 25825	79
0.97045 71520	2.45968 00304	0.12654 25123	85 59	2.82126 25499	78
0.96540 27800	2.44796 55051	0.13695 55734	85 38	2.78509 25172	77
0.95996 84748	2.43541 15773	0.14733 49785	85 16	2.74892 24845	76
0.95415 96125	2.42201 41749	0.15767 73727	84 54	2.71275 24518	75
0.94798 22318	2.40770 13262	0.16797 92308	84 32	2.67658 24191	74
0.94144 22181	2.39276 21449	0.17824 67907	84 9	2.64041 23864	73
0.93454 60808	2.37694 67187	0.18841 61360	83 45	2.60424 23537	72
0.92730 05843	2.36036 63252	0.19860 30778	83 21	2.56807 23210	71
0.91971 27230	2.34304 20970	0.20870 31860	82 57	2.53190 22883	70
0.91178 07080	2.32490 08377	0.21874 17592	82 32	2.49573 22556	69
0.90353 93117	2.30626 08161	0.22871 38038	82 7	2.45956 22230	68
0.89496 91497	2.28685 07750	0.23861 40135	81 41	2.42339 21903	67
0.88608 71836	2.26670 53047	0.24843 67407	81 14	2.38722 21576	66
0.87690 16690	2.24612 10260	0.25817 59833	80 47	2.35105 21249	65
0.86742 09743	2.22485 40364	0.26784 53494	80 19	2.31488 20922	64
0.85765 30425	2.20302 49697	0.27737 80358	79 50	2.27871 20595	63
0.84760 83633	2.18065 96824	0.28682 68001	79 20	2.24254 20268	62
0.83729 40541	2.15778 81197	0.29610 30332	78 50	2.20637 19941	61
0.82671 93416	2.13441 00706	0.30538 12272	78 19	2.17020 19614	60
0.81589 35129	2.11061 87227	0.31446 99478	77 47	2.13403 19287	59
0.80482 86467	2.08641 87672	0.32342 08014	77 14	2.09786 18960	58
0.79352 47015	2.06184 13420	0.33222 30026	76 40	2.06169 18634	57
0.78200 01643	2.03689 38602	0.34086 87415	76 5	2.02552 18307	56
0.77026 00411	2.01162 85056	0.34934 41494	75 29	1.98935 17980	55
0.75831 63101	1.98606 70958	0.35763 82641	74 53	1.95318 17653	54
0.74617 84612	1.96028 34320	0.36573 84071	74 14	1.91701 17326	53
0.73384 75030	1.93421 50843	0.37363 14953	73 35	1.88084 16999	52
0.72134 15696	1.90798 84771	0.38130 31100	72 55	1.84467 16672	51
0.70866 64787	1.88159 71433	0.38873 83616	72 13	1.80850 16345	50
0.69583 13178	1.85508 32817	0.39592 14068	71 30	1.77233 16018	49
0.68284 48256	1.82847 67117	0.40283 85070	70 46	1.73616 15691	48
0.66971 56781	1.80181 03311	0.40946 30040	70 1	1.69999 15365	47
0.65645 24120	1.77511 60734	0.41578 52840	69 14	1.66382 15038	46
0.64306 34108	1.74842 03062	0.42178 27675	68 25	1.62765 14711	45
A(r)	D(r)	E(r)	ϕ	Fϕ	r

K = 3.3698680207, K' = 1.5784865777, E = 1.0278430920, E' = 1.5620622205,

r	Fφ	φ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0'	0.00000 00000	1.00000 00000	0.00000 00000
1	0.03744 29781	2	9	[illegible]	[illegible]	0.01396 87846
2	0.07488 59561	4	17	[illegible]	[illegible]	0.02793 06081
3	0.11232 89342	6	26	[illegible]	[illegible]	0.04191 44020
4	0.14977 19123	8	35	[illegible]	[illegible]	0.05589 54231
5	0.18721 48904	10	40	[illegible]	[illegible]	0.06988 43359
6	0.22465 78684	12	46	[illegible]	[illegible]	0.08388 30056
7	0.26210 08465	14	51	[illegible]	[illegible]	0.09789 34813
8	0.29954 38246	16	55	[illegible]	[illegible]	0.11191 71000
9	0.33698 68027	18	58	[illegible]	[illegible]	0.12595 57152
10	0.37442 97807	20	59	[illegible]	[illegible]	0.14001 05412
11	0.41187 27588	22	58	[illegible]	[illegible]	0.15408 29167
12	0.44931 57369	24	56	[illegible]	[illegible]	0.16817 39151
13	0.48675 87150	26	52	[illegible]	[illegible]	0.18228 45483
14	0.52420 16930	28	46	[illegible]	[illegible]	0.19641 55524
15	0.56164 46711	30	38	[illegible]	[illegible]	0.21056 71740
16	0.59908 76492	32	28	[illegible]	[illegible]	0.22474 00071
17	0.63653 06273	34	16	[illegible]	[illegible]	[illegible]
18	0.67397 36053	36	2	[illegible]	[illegible]	[illegible]
19	0.71141 65834	37	46	[illegible]	[illegible]	[illegible]
20	0.74885 95615	39	27	[illegible]	[illegible]	[illegible]
21	0.78630 25396	41	6	[illegible]	[illegible]	[illegible]
22	0.82374 55176	42	42	[illegible]	[illegible]	[illegible]
23	0.86118 84957	44	16	[illegible]	[illegible]	[illegible]
24	0.89863 14738	45	48	[illegible]	[illegible]	[illegible]
25	0.93607 44519	47	18	[illegible]	[illegible]	[illegible]
26	0.97351 74299	48	45	[illegible]	[illegible]	[illegible]
27	1.01096 04080	50	10	[illegible]	[illegible]	[illegible]
28	1.04840 33861	51	32	[illegible]	[illegible]	[illegible]
29	1.08584 63641	52	52	[illegible]	[illegible]	[illegible]
30	1.12328 93422	54	10	[illegible]	[illegible]	[illegible]
31	1.16073 23203	55	26	[illegible]	[illegible]	[illegible]
32	1.19817 52984	56	39	[illegible]	[illegible]	[illegible]
33	1.23561 82764	57	50	[illegible]	[illegible]	[illegible]
34	1.27306 12545	59	0	[illegible]	[illegible]	[illegible]
35	1.31050 42326	60	7	[illegible]	[illegible]	[illegible]
36	1.34794 72107	61	12	[illegible]	[illegible]	[illegible]
37	1.38539 01887	62	15	[illegible]	[illegible]	[illegible]
38	1.42283 31668	63	16	[illegible]	[illegible]	[illegible]
39	1.46027 61449	64	15	[illegible]	[illegible]	[illegible]
40	1.49771 91230	65	12	[illegible]	[illegible]	[illegible]
41	1.53516 21010	66	7	[illegible]	[illegible]	[illegible]
42	1.57260 50791	67	1	[illegible]	[illegible]	[illegible]
43	1.61004 80572	67	54	[illegible]	[illegible]	[illegible]
44	1.64749 10353	68	44	[illegible]	[illegible]	[illegible]
45	1.68493 40133	69	32	[illegible]	[illegible]	[illegible]
90 r	K	[illegible]		[illegible]	[illegible]	[illegible]

$q = 0.220567160881104$, $\Theta 0 = 0.6404169465$, $\mathrm{HK} = 1.4675481002$

B(r)	C(r)	G(r)	ψ	Fψ	90-r
1.00000 00000	2.68054 04437	0.00000 00000	90° 0′	3.36986 80267	90
0.99978 62112	2.68000 30787	0.01069 40135	89 42	3.33242 50486	89
0.99914 50809	2.67839 44283	0.02138 78301	89 24	3.29498 20705	88
0.99807 73170	2.67571 48255	0.03207 07423	89 6	3.25753 90925	87
0.99658 40072	2.67196 85860	0.04275 96200	88 48	3.22009 61144	86
0.99466 70666	2.66716 00043	0.05344 44040	88 30	3.18265 31363	85
0.99232 83334	2.66129 84418	0.06409 89867	88 12	3.14521 01582	84
0.98957 04045	2.65438 04156	0.07475 12085	87 53	3.10776 71802	83
0.98639 64786	2.64644 30842	0.08538 88428	87 35	3.07032 42021	82
0.98280 98400	2.63747 07206	0.09600 95847	87 16	3.03288 12240	81
0.97881 44197	2.62748 46381	0.10661 10385	86 57	2.99543 82459	80
0.97441 46407	2.61649 85778	0.11719 07054	86 37	2.95799 52679	79
0.96961 51474	2.60452 70741	0.12774 59701	86 18	2.92055 22898	78
0.96442 11348	2.59158 83828	0.13827 40870	85 58	2.88310 93117	77
0.95883 81466	2.57769 84666	0.14877 21662	85 38	2.84566 63336	76
0.95287 21117	2.56287 60342	0.15923 71880	85 17	2.80822 33556	75
0.94652 93261	2.54714 40664	0.16966 58376	84 56	2.77078 03775	74
0.93981 64141	2.53052 13208	0.18005 47885	84 35	2.73333 73994	73
0.93274 01149	2.51303 13218	0.19040 03840	84 13	2.69589 44213	72
0.92530 80410	2.49469 78294	0.20069 87730	83 51	2.65845 14433	71
0.91752 80534	2.47554 56698	0.21094 58556	83 28	2.62100 84652	70
0.90940 84726	2.45560 07207	0.22113 72633	83 5	2.58356 54871	69
0.90095 80854	2.43488 09556	0.23126 83422	82 41	2.54612 25090	68
0.89217 98975	2.41341 06085	0.24133 41265	82 16	2.50867 95310	67
0.88308 87660	2.39123 25787	0.25132 93157	81 51	2.47123 65529	66
0.87369 14666	2.36814 44841	0.26124 82801	81 25	2.43379 35748	65
0.86399 79175	2.34498 78070	0.27108 48837	80 59	2.39635 05967	64
0.85401 47452	2.32085 13053	0.28083 27574	80 32	2.35890 76187	63
0.84375 39427	2.29615 02414	0.29048 49692	80 4	2.32146 46406	62
0.83322 11855	2.27091 33305	0.30003 41444	79 35	2.28402 16625	61
0.82243 90060	2.24514 67182	0.30947 24031	79 5	2.24657 86844	60
0.81139 63227	2.21889 30062	0.31879 13270	78 35	2.20913 57064	59
0.80011 75798	2.19218 05710	0.32798 10272	78 4	2.17169 27283	58
0.78860 89149	2.16506 18621	0.33703 46027	77 31	2.13424 97502	57
0.77688 00911	2.13755 39706	0.34593 91087	76 58	2.09680 67721	56
0.76494 09778	2.10969 82742	0.35468 48152	76 23	2.05936 37941	55
0.75280 11118	2.08153 01123	0.36325 91686	75 48	2.02192 08160	54
0.74047 12758	2.05308 63586	0.37165 06805	75 11	1.98447 78379	53
0.72796 02806	2.02440 22011	0.37984 57377	74 34	1.94703 48599	52
0.71527 81414	1.99551 41474	0.38783 03601	73 55	1.90959 18818	51
0.70243 44746	1.96645 88118	0.39558 05506	73 14	1.87214 89037	50
0.68943 87618	1.93727 11624	0.40310 74491	72 33	1.83470 59256	49
0.67630 03566	1.90799 18345	0.41036 71725	71 50	1.79726 29476	48
0.66302 88617	1.87864 98345	0.41735 08655	71 6	1.75981 99695	47
0.64963 34566	1.84928 72824	0.42403 96200	70 20	1.72237 69914	46
0.63612 06340	1.81993 84164	0.43041 34495	69 32	1.68493 40133	45
[illegible]	[illegible]	[illegible]	φ	Fφ	r

K = 3.6004224992, K′ = 1.5766779816, E = 1.022312588, E′ = 1.5049475630,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.03889 35833	2 14	0.02751 52459	1.00053 54142	0.01357 81428
2	0.07778 71666	4 27	0.05491 49171	1.00214 11230	0.02715 91204
3	0.11668 07500	6 40	0.08208 48196	1.00481 55243	0.04074 57840
4	0.15557 43333	8 53	0.10891 34862	1.00855 59486	0.05434 08022
5	0.19446 79166	11 4	0.13529 34531	1.01335 86590	0.06794 71815
6	0.23336 14999	13 15	0.16112 24388	1.01921 88518	0.08156 73027
7	0.27225 50833	15 25	0.18630 43989	1.02613 06577	0.09520 38101
8	0.31114 86666	17 33	0.21075 04315	1.03408 71422	0.10885 91438
9	0.35004 22499	19 40	0.23437 95237	1.04308 03072	0.12253 56111
10	0.38893 58332	21 45	0.25711 91248	1.05310 10924	0.13623 53681
11	0.42782 94166	23 48	0.27890 55463	1.06413 93774	0.14996 04030
12	0.46672 29999	25 50	0.29968 41874	1.07618 30836	0.16371 25182
13	0.50561 65832	27 50	0.31940 95974	1.08922 20769	0.17749 33141
14	0.54451 01665	29 47	0.33804 53836	1.10324 21710	0.19130 41733
15	0.58340 37499	31 42	0.35556 39822	1.11822 81308	0.20514 62446
16	0.62229 73332	33 35	0.37194 63079	1.13410 51764	0.21902 04287
17	0.66119 09165	35 26	0.38718 13038	1.15103 68883	0.23292 73037
18	0.70008 44998	37 14	0.40126 54102	1.16882 58124	0.24686 74120
19	0.73897 80832	38 59	0.41420 19722	1.18751 34668	0.26084 06476
20	0.77787 16665	40 42	0.42600 06064	1.20708 03483	0.27484 68440
21	0.81676 52498	42 23	0.43667 65427	1.22750 50404	0.28888 54037
22	0.85565 88331	44 1	0.44624 99581	1.24876 87226	0.30295 50475
23	0.89455 24165	45 37	0.45474 53170	1.27084 01798	0.31705 62057
24	0.93344 59998	47 10	0.46219 07281	1.29371 48135	0.33118 56005
25	0.97233 95831	48 40	0.46861 73287	1.31735 01537	0.34534 10830
26	1.01123 31664	50 8	0.47405 87042	1.34172 07728	0.35952 31012
27	1.05012 67498	51 33	0.47855 03463	1.36681 82944	0.37372 63757
28	1.08902 03331	52 56	0.48212 91569	1.39259 74348	0.38794 88503
29	1.12791 39164	54 17	0.48483 29959	1.41903 59703	0.40218 72381
30	1.16680 74997	55 35	0.48670 02770	1.44610 48057	0.41643 78306
31	1.20570 10830	56 50	0.48776 96603	1.47377 39701	0.43069 65861
32	1.24459 46664	58 4	0.48807 94838	1.50201 26433	0.44495 90840
33	1.28348 82497	59 14	0.48766 80032	1.53078 91792	0.45922 05300
34	1.32238 18330	60 23	0.48657 26520	1.56007 11317	0.47347 57948
35	1.36127 54163	61 30	0.48483 01030	1.58982 52804	0.48771 93356
36	1.40016 89997	62 34	0.48247 66647	1.62001 70598	0.50194 52865
37	1.43906 25830	63 36	0.47954 51456	1.65061 35805	0.51614 74196
38	1.47795 61663	64 36	0.47607 07644	1.68157 77058	0.53031 91603
39	1.51684 97496	65 35	0.47208 50753	1.71287 39955	0.54445 35952
40	1.55574 33330	66 31	0.46761 89121	1.74446 58318	0.55854 34803
41	1.59463 69163	67 25	0.46270 17621	1.77631 60110	0.57258 12511
42	1.63353 04996	68 18	0.45736 17475	1.80838 67918	0.58655 90333
43	1.67242 40829	69 9	0.45162 56249	1.84063 99362	0.60046 86540
44	1.71131 76663	69 58	0.44551 87962	1.87303 67513	0.61430 16540
45	1.75021 12496	70 45	0.43906 53283	1.90553 81344	0.62804 93057
90−r	Fψ	ψ	G(r)	C(r)	B(r)

= 0.242012074300066, OO = 0.6211317406, HK = 1.4872214813

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	2.86152 59727	0.00000 00000	90° 0′	3.50042 24992	90
0.99977 01249	2.86302 51580	0.01078 10089	89 44	3.46152 89158	89
0.99911 07583	2.86212 47052	0.02156 04536	89 27	3.42263 53325	88
0.99801 30755	2.85912 64401	0.03233 63507	89 11	3.38374 17492	87
0.99647 11670	2.85403 17485	0.04310 70526	88 55	3.34484 81659	86
0.99449 10348	2.84685 56077	0.05387 07471	88 38	3.30595 45826	85
0.99207 55974	2.84290 60456	0.06462 56168	88 21	3.26706 09992	84
0.98922 70307	2.83526 71062	0.07536 07836	88 5	3.22816 74159	83
0.98595 01884	2.82647 70377	0.08610 13069	87 48	3.18927 38326	82
0.98224 70350	2.81644 16722	0.09681 81718	87 30	3.15038 02493	81
0.97812 42473	2.80517 21517	0.10751 82770	87 13	3.11148 66659	80
0.97358 41028	2.79265 56010	0.11819 94268	86 55	3.07259 30826	79
0.96863 28755	2.77819 05543	0.12885 93997	86 37	3.03369 94993	78
0.96327 06226	2.76204 10042	0.13949 54938	86 19	2.99480 59160	77
0.95751 06711	2.74030 32957	0.15010 54088	86 1	2.95591 23326	76
0.95137 03036	2.73493 54142	0.16068 63418	85 42	2.91701 87493	75
0.94483 18022	2.71545 13165	0.17123 58724	85 23	2.87812 51660	74
0.93792 01329	2.69677 55303	0.18174 94560	85 3	2.83923 15827	73
0.93063 48270	2.67223 47307	0.19222 53007	84 43	2.80033 79993	72
0.92298 50601	2.65625 20730	0.20265 94294	84 22	2.76144 44160	71
0.91498 21585	2.64536 14041	0.21304 86001	84 1	2.72255 08327	70
0.90663 20234	2.61303 30698	0.22339 72050	83 39	2.68365 72494	69
0.89791 44068	2.59619 50797	0.23367 27719	83 17	2.64476 36660	68
0.88892 80751	2.56602 22648	0.24389 00414	82 54	2.60587 00827	67
0.87959 40054	2.54129 27073	0.25406 39681	82 31	2.56697 64994	66
0.86995 02960	2.51581 03430	0.26415 93822	82 7	2.52808 29161	65
0.86001 72861	2.49601 06190	0.27418 07525	81 42	2.48918 93327	64
0.84977 49195	2.46722 43050	0.28412 19570	81 16	2.45029 57494	63
0.83926 34134	2.43510 23702	0.29397 06053	80 50	2.41140 21661	62
0.82848 32567	2.40204 07447	0.30373 74301	80 23	2.37250 85828	61
0.81744 47352	2.37644 38854	0.31340 72593	79 55	2.33361 49994	60
0.80615 84718	2.34965 25018	0.32294 70773	79 26	2.29472 14161	59
0.79464 59437	2.30434 50413	0.33246 06873	78 56	2.25582 78328	58
0.78288 80860	2.28513 05230	0.34168 70724	78 26	2.21693 42495	57
0.77091 92160	2.25590 87384	0.35095 03539	77 54	2.17804 06662	56
0.75871 81476	2.22730 31151	0.35987 82486	77 21	2.13914 70828	55
0.74638 25648	2.19616 26068	0.36874 14137	76 47	2.10025 34995	54
0.73383 28587	2.16452 35768	0.37743 37807	76 12	2.06135 99162	53
0.72110 07344	2.13263 67768	0.38594 22578	75 36	2.02246 63329	52
0.70822 38803	2.10061 27578	0.39428 30813	74 58	1.98357 27495	51
0.69518 46210	2.06823 31126	0.40238 14155	74 20	1.94467 91662	50
0.68200 31247	2.03570 88131	0.41032 14630	73 40	1.90578 55829	49
0.66868 00678	2.00327 33780	0.41784 63843	72 58	1.86689 19996	48
0.65525 23649	1.97080 64170	0.42520 82470	72 16	1.82799 84162	47
0.64170 26158	1.93810 40179	0.43228 70822	71 31	1.78910 48329	46
0.62804 03087	1.90583 81344	0.43906 53283	70 45	1.75021 12496	45
A(r)	D(r)	E(r)	φ	Fφ	r

K = 3.6518559695, K′ = 1.5751136078, E = 1.017236918, E′ = 1.50049679

r	$F\phi$	ϕ		E(r)	D(r)	A(r)
0	0.00000 00000	0°	0′	0.00000 00000	1.00000 00000	0.00000 0000
1	0.04057 61774	2	1	0.02925 15342	1.00059 38572	0.01311 0258
2	0.08115 23549	4	29	0.05837 13484	1.00237 48641	0.02624 2297
3	0.12172 85323	6	55	0.08722 94380	1.00534 13202	0.03937 2874
4	0.16230 47098	9	16	0.11569 91812	1.00949 04102	0.05251 4706
5	0.20288 08872	11	33	0.14365 80152	1.01481 81886	0.06567 1442
6	0.24345 70646	13	49	0.17099 33783	1.02131 95401	0.07884 66485
7	0.28403 32421	16	4	0.19759 49853	1.02898 82841	0.09204 37819
8	0.32460 94195	18	17	0.22336 49075	1.03781 70450	0.10526 61731
9	0.36518 55969	20	29	0.24821 30381	1.04779 73504	0.11851 70041
10	0.40576 17744	22	39	0.27206 31341	1.05891 95857	0.13179 09889
11	0.44633 79518	24	46	0.29484 42309	1.07117 30024	0.14511 88534
12	0.48691 41293	26	52	0.31649 08365	1.08454 87174	0.15846 93168
13	0.52749 03067	28	56	0.33698 34178	1.09902 47131	0.17186 20796
14	0.56806 64841	30	58	0.35625 00959	1.11459 58374	0.18529 02711
15	0.60864 26616	32	55	0.37430 12782	1.13124 38038	0.19877 38016
16	0.64921 88390	34	51	0.39109 44430	1.14895 21048	0.21229 62758
17	0.68979 50165	36	44	0.40663 10147	1.16770 34514	0.22586 50123
18	0.73037 11939	38	30	0.42091 36481	1.18747 88983	0.23948 10211
19	0.77094 73713	40	24	0.43395 14533	1.20825 87235	0.25314 40894
20	0.81152 35488	42	9	0.44576 06829	1.23002 19929	0.26685 72683
21	0.85209 97262	43	51	0.45636 30044	1.25274 66524	0.28061 78690
22	0.89267 59037	45	31	0.46578 70783	1.27640 95335	0.29442 94067
23	0.93325 20811	47	8	0.47406 47564	1.30098 63590	0.30828 21794
24	0.97382 82585	48	42	0.48123 03147	1.32645 17509	0.32218 40690
25	1.01440 44360	50	13	0.48732 27312	1.35277 92303	0.33613 05773
26	1.05498 06134	51	42	0.49238 26189	1.37994 12721	0.35011 08097
27	1.09555 67908	53	8	0.49645 21069	1.40790 92268	0.36414 94689
28	1.13613 29683	54	31	0.49957 47663	1.43665 34239	0.37821 68497
29	1.17670 91457	55	51	0.50179 41807	1.46614 31412	0.39231 88350
30	1.21728 53232	57	9	0.50318 44701	1.49634 66307	0.40645 18927
31	1.25786 15006	58	25	0.50369 93739	1.52723 11369	0.42061 20743
32	1.29843 76780	59	38	0.50347 21104	1.55876 29167	0.43479 50141
33	1.33901 38555	60	48	0.50251 50624	1.59090 72622	0.44899 59393
34	1.37959 00329	61	56	0.50086 95051	1.62362 85541	0.46320 96265
35	1.42016 62104	63	2	0.49857 57270	1.65689 01387	0.47743 09032
36	1.46074 23878	64	5	0.49567 22903	1.69065 46858	0.49165 45218
37	1.50131 85652	65	7	0.49219 65269	1.72488 37696	0.50586 92998
38	1.54189 47427	66	6	0.48818 41583	1.75953 83514	0.52007 39919
39	1.58247 09201	67	3	0.48366 93168	1.79457 84847	0.53425 94285
40	1.62304 70975	67	58	0.47868 48999	1.82996 35024	0.54841 80268
41	1.66362 32750	68	51	0.47326 96189	1.86565 20265	0.56254 18461
42	1.70419 94524	69	42	0.46742 69071	1.90160 21899	0.57662 25993
43	1.74477 56299	70	31	0.46121 10428	1.93777 07867	0.59065 16209
44	1.78535 18073	71	19	0.45463 91336	1.97411 80881	0.60461 99794
45	1.82592 79847	72	5	0.44773 57684	2.01059 11817	0.61851 83573
90-r	$F\psi$	ψ		G(r)		

TABLE $\theta = 84°$

$q = 0.257940195766337$, $\Theta 0 = 0.4929628191$, HK = 1.5205617314

B(r)	C(r)	G(r)	ψ	Fψ	90-r
1.00000 00000	3.09301 99213	0.00000 00000	90° 0′	3.65185 59695	90
0.99977 07150	3.09233 85676	0.01085 90483	89 45	3.61127 97920	89
0.99908 31458	3.09029 54977	0.02171 66503	89 31	3.57070 36146	88
0.99793 81489	3.08689 36827	0.03257 13506	89 16	3.53012 74372	87
0.99633 71496	3.08213 80679	0.04342 16747	89 1	3.48955 12597	86
0.99428 21381	3.07603 55627	0.05426 61204	88 47	3.44897 50823	85
0.99177 56649	3.06859 50269	0.06510 31473	88 32	3.40839 89048	84
0.98882 08340	3.05982 72527	0.07593 11673	88 17	3.36782 27274	83
0.98542 12055	3.04974 49431	0.08674 85345	88 2	3.32724 65500	82
0.98158 12363	3.03836 26866	0.09755 35344	87 46	3.28667 03725	81
0.97730 53698	3.02569 69280	0.10834 43731	87 30	3.24609 41951	80
0.97259 89240	3.01176 59358	0.11911 91660	87 14	3.20551 80177	79
0.96746 76286	2.99658 97659	0.12987 59255	86 58	3.16494 18402	78
0.96191 77007	2.98019 02223	0.14061 25487	86 42	3.12436 56628	77
0.95595 58299	2.96259 08137	0.15132 68040	86 25	3.08378 94853	76
0.94958 91609	2.94381 67083	0.16201 63172	86 8	3.04321 33079	75
0.94282 52769	2.92389 46843	0.17267 85562	85 50	3.00263 71305	74
0.93567 21802	2.90285 30783	0.18331 08161	85 32	2.96206 09530	73
0.92813 82732	2.88072 17308	0.19391 02013	85 14	2.92148 47756	72
0.92023 23370	2.85753 19293	0.20447 36088	84 55	2.88090 85981	71
0.91196 35133	2.83331 63402	0.21499 77081	84 36	2.84033 24207	70
0.90334 12763	2.80810 89017	0.22547 89218	84 16	2.79975 62433	69
0.89437 54154	2.78194 51210	0.23591 34034	83 55	2.75918 00658	68
0.88507 60096	2.75486 11988	0.24629 70143	83 34	2.71860 38884	67
0.87545 34034	2.72689 48173	0.25662 52995	83 13	2.67802 77109	66
0.86551 81826	2.69808 46313	0.26689 34606	82 51	2.63745 15335	65
0.85528 11491	2.66847 02880	0.27709 63287	82 28	2.59687 53561	64
0.84475 32958	2.63809 23575	0.28722 83335	82 4	2.55629 91786	63
0.83394 57809	2.60699 22004	0.29728 34722	81 39	2.51572 30012	62
0.82286 99019	2.57521 21966	0.30725 52753	81 14	2.47514 68238	61
0.81153 70701	2.54279 50725	0.31713 67705	80 48	2.43457 06463	60
0.79995 87840	2.50978 44281	0.32692 04449	80 21	2.39399 44689	59
0.78814 66036	2.47622 43648	0.33659 82039	79 53	2.35341 82914	58
0.77611 21247	2.44215 94723	0.34616 13287	79 24	2.31284 21140	57
0.76386 69524	2.40763 47564	0.35560 04313	78 54	2.27226 59366	56
0.75142 26764	2.37269 55671	0.36490 54063	78 23	2.23168 97591	55
0.73879 08451	2.33738 75276	0.37406 53814	77 51	2.19111 35817	54
0.72598 29409	2.30175 64635	0.38306 86651	77 18	2.15053 74042	53
0.71301 03561	2.26584 83337	0.39190 26919	76 44	2.10996 12268	52
0.69988 43682	2.22970 91619	0.40055 39659	76 8	2.06938 50494	51
0.68661 61172	2.19338 49695	0.40900 80023	75 31	2.02880 88719	50
0.67321 65825	2.15692 17102	0.41724 92673	74 53	1.98823 26945	49
0.65969 65607	2.12036 52053	0.42526 11165	74 13	1.94765 65171	48
0.64606 66446	2.08376 10820	0.43302 57335	73 32	1.90708 03396	47
0.63233 72022	2.04715 47117	0.44052 40667	72 49	1.86650 41622	46
0.61851 83573	2.01059 11517	0.44773 57684	72 5	1.82592 79847	45
[illegible]	D(r)	E(r)	φ	Fφ	r

K = 3.8317419998, K′ = 1.5737921309, E = 1.0126635062, E′ = 1.5678000740,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.04257 49111	2 26	0.03129 75841	1.00066 67396	0.01256 98450
2	0.08514 98222	4 52	0.06244 25476	1.00266 63052	0.02514 45765
3	0.12772 47333	7 18	0.09328 44601	1.00599 70074	0.03772 90570
4	0.17029 96444	9 43	0.12367 72052	1.01065 59692	0.05032 81006
5	0.21287 45555	12 6	0.15348 00740	1.01663 88247	0.06294 64495
6	0.25544 94667	14 29	0.18256 40780	1.02394 03165	0.07558 87497
7	0.29802 43778	16 50	0.21080 45154	1.03255 39030	0.08825 95281
8	0.34059 92889	19 9	0.23800 12866	1.04247 18453	0.10096 31685
9	0.38317 42000	21 26	0.26432 54039	1.05368 52030	0.11370 38805
10	0.42574 91111	23 42	0.28942 06026	1.06618 38299	0.12648 57214
11	0.46832 40222	25 55	0.31330 37505	1.07995 63700	0.13931 24846
12	0.51089 89333	28 5	0.33591 49667	1.09499 02519	0.15218 77082
13	0.55347 38444	30 13	0.35720 74739	1.11127 16844	0.16511 49087
14	0.59604 87555	32 18	0.37714 72117	1.12878 56513	0.17809 69700
15	0.63862 36666	34 21	0.39571 22464	1.14751 59963	0.19113 67239
16	0.68119 85777	36 20	0.41289 20138	1.16744 49685	0.20423 66315
17	0.72377 34889	38 17	0.42868 64336	1.18855 41178	0.21739 88246
18	0.76634 84000	40 11	0.44310 49337	1.21082 33907	0.23062 50891
19	0.80892 33111	42 1	0.45616 54173	1.23423 15771	0.24391 68485
20	0.85149 82222	43 49	0.46789 32075	1.25875 62174	0.25727 81484
21	0.89407 31333	45 33	0.47831 09952	1.28437 36007	0.27070 06428
22	0.93664 80444	47 15	0.48748 28142	1.31105 87034	0.28419 35800
23	0.97922 29555	48 53	0.49542 30625	1.33878 54900	0.29775 37010
24	1.02179 78666	50 28	0.50218 55842	1.36752 63142	0.31138 06778
25	1.06437 27777	52 0	0.50781 78217	1.39725 28218	0.32507 32040
26	1.10694 76888	53 29	0.51236 90454	1.42793 41552	0.33882 98857
27	1.14952 25999	54 56	0.51588 96635	1.45954 00105	0.35264 87830
28	1.19209 75110	56 19	0.51843 06138	1.49203 76904	0.36652 74084
29	1.23467 24222	57 39	0.52004 28338	1.52539 35243	0.38046 31019
30	1.27724 73333	58 50	0.52077 68087	1.55957 26706	0.39445 24378
31	1.31982 22444	60 13	0.52068 21896	1.59453 90851	0.40849 15164
32	1.36239 71555	61 24	0.51980 74709	1.63025 55479	0.42257 61140
33	1.40497 20666	62 34	0.51819 97811	1.66668 30814	0.43670 14735
34	1.44754 69777	63 41	0.51590 45944	1.70378 30728	0.45086 23088
35	1.49012 18888	64 46	0.51296 56697	1.74151 57980	0.46505 30026
36	1.53269 67999	65 48	0.50942 48984	1.77983 74487	0.47926 74999
37	1.57527 17110	66 48	0.50532 22421	1.81870 61627	0.49349 89386
38	1.61784 66221	67 46	0.50069 86036	1.85807 81564	0.50774 03615
39	1.66042 15332	68 41	0.49558 12646	1.89790 86607	0.52198 42419
40	1.70299 64444	69 35	0.49001 29952	1.93815 19509	0.53622 26281
41	1.74557 13555	70 26	0.48402 29834	1.97876 14331	0.55044 71457
42	1.78814 62666	71 16	0.47764 14227	2.01968 95998	0.56464 98899
43	1.83072 11777	72 3	0.47089 66670	2.06088 81669	0.57881 99394
44	1.87329 60888	72 49	0.46381 52836	2.10230 80805	0.59294 76712
45	1.91587 09999	73 33	0.45642 21286	2.14389 93702	0.60702 49768
90 r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.275170801873563$, $\Theta 0 = 0.4610905222$, $HK = 1.5588714533$

B(r)	C(r)	G(r)	ψ		Fψ	90−r
1.00000 00000	3.38728 70037	0.00000 00000	90°	0′	3.83174 19998	90
0.99976 05011	3.38640 99904	0.01092 82185	89	47	3.78916 70887	89
0.99904 23353	3.38413 05337	0.02185 52713	89	34	3.74659 21776	88
0.99784 64504	3.38020 28815	0.03277 99847	89	22	3.70401 72665	87
0.99617 44409	3.37470 40379	0.04370 11679	89	9	3.66144 23554	86
0.99402 85290	3.36764 82512	0.05461 76051	88	56	3.61886 74443	85
0.99141 15622	3.35904 06061	0.06552 80467	88	43	3.57629 25331	84
0.98832 70058	3.34891 04507	0.07643 12000	88	29	3.53371 76220	83
0.98477 80335	3.33728 64694	0.08732 57205	88	16	3.49114 27109	82
0.98077 20177	3.32410 15504	0.09821 02023	88	2	3.44856 77998	81
0.97631 15168	3.30946 52989	0.10908 31677	87	49	3.40599 28887	80
0.97140 32610	3.29336 94854	0.11994 30573	87	35	3.36341 79776	79
0.96605 30420	3.27583 70099	0.13078 82183	87	20	3.32084 30665	78
0.96026 05874	3.25689 68018	0.14161 68937	87	6	3.27826 81554	77
0.95403 85520	3.23657 38054	0.15242 72092	86	51	3.23569 32443	76
0.94742 84047	3.21489 01220	0.16321 71605	86	35	3.19311 83332	75
0.94038 78686	3.19190 43978	0.17398 45990	86	20	3.15054 34221	74
0.93294 55400	3.16762 33486	0.18472 72171	86	4	3.10796 85109	73
0.92511 00158	3.14209 13099	0.19544 25321	85	48	3.06539 35998	72
0.91689 37204	3.11534 56304	0.20612 78689	85	31	3.02281 86887	71
0.90830 41205	3.08742 47870	0.21678 03410	85	13	2.98024 37776	70
0.89935 26403	3.05836 91177	0.22739 68349	84	55	2.93766 88665	69
0.89005 01452	3.02822 03368	0.23797 39802	84	37	2.89509 39554	68
0.88040 78152	2.99702 15345	0.24850 81357	84	18	2.85251 90443	67
0.87043 71170	2.96481 70025	0.25899 53603	83	58	2.80994 41332	66
0.86014 97763	2.93165 45095	0.26943 13876	83	38	2.76736 92221	65
0.84955 77491	2.89757 47041	0.27981 15077	83	17	2.72479 43110	64
0.83867 31042	2.86264 13272	0.29013 00871	82	55	2.68221 93999	63
0.82750 84183	2.82687 09732	0.30038 41353	82	33	2.63964 44888	62
0.81607 30570	2.79031 32412	0.31056 81708	82	10	2.59706 95776	61
0.80438 83372	2.75300 84151	0.32066 77330	81	46	2.55449 46665	60
0.79245 82474	2.71518 75345	0.33068 49323	81	21	2.51191 97554	59
0.78029 84129	2.67666 21047	0.34060 93073	80	55	2.46934 48443	58
0.76792 15844	2.63757 42081	0.35043 27789	80	28	2.42676 99332	57
0.75534 05043	2.59797 03158	0.36014 66018	80	0	2.38419 50221	56
0.74256 78883	2.55792 12108	0.36974 13124	79	31	2.34162 01110	55
0.72961 64864	2.51746 10471	0.37920 66740	79	2	2.29904 51999	54
0.71649 88603	2.47665 16742	0.38853 16185	78	30	2.25647 02888	53
0.70322 68545	2.43554 36438	0.39770 41848	77	58	2.21389 53777	52
0.68981 48629	2.39419 10827	0.40671 14546	77	24	2.17132 04666	51
0.67627 08370	2.35264 71220	0.41553 94843	76	50	2.12874 55554	50
0.66261 05010	2.31096 47190	0.42417 32345	76	13	2.08617 06443	49
0.64884 48467	2.26919 65810	0.43259 64967	75	35	2.04359 57332	48
0.63498 41780	2.22739 50055	0.44079 18172	74	56	2.00102 08221	47
0.62104 06860	2.18561 22515	0.44874 04204	74	16	1.95844 59110	46
0.60702 40768	2.14389 95702	0.45642 21286	73	33	1.91587 09999	45
A(r)	D(r)	E(r)	ϕ		Fϕ	r

K = 4.0527581695, K′ = 1.5727124350, E = 1.0080479569, E′ = 1.5688837196,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.04503 06463	2 35	0.03379 31823	1.00070 14018	0.01180 42847
2	0.09006 12927	5 9	0.06740 53633	1.00301 53071	0.02370 47003
3	0.13509 19390	7 43	0.10065 84494	1.00684 07791	0.03570 77106
4	0.18012 25853	10 16	0.13338 06630	1.01217 16068	0.04763 01858
5	0.22515 32316	12 48	0.16540 61602	1.01900 67332	0.05959 52742
6	0.27018 38780	15 18	0.19658 33730	1.02734 94450	0.07158 10286
7	0.31521 45243	17 46	0.22677 10168	1.03710 30291	0.08360 40670
8	0.36024 51706	20 13	0.25584 26948	1.04852 91558	0.09567 00478
9	0.40527 58170	22 37	0.28368 78091	1.06131 94387	0.10778 20441
10	0.45030 64633	24 58	0.31021 07894	1.07561 24197	0.11994 80182
11	0.49533 71096	27 18	0.33533 45137	1.09139 05585	0.13217 11972
12	0.54036 77559	29 34	0.35899 71966	1.10859 87266	0.14445 69185
13	0.58539 84023	31 47	0.38115 35291	1.12723 44947	0.15680 07563
14	0.63042 90486	33 57	0.40177 30714	1.14728 80243	0.16923 37958
15	0.67545 96949	36 4	0.42084 23033	1.16874 23030	0.18173 29260
16	0.72049 03413	38 8	0.43835 74800	1.19157 88591	0.19431 06684
17	0.76552 09876	40 8	0.45432 03518	1.21577 75016	0.20697 06661
18	0.81055 16339	42 5	0.46877 87966	1.24131 81358	0.21971 30408
19	0.85558 22802	43 58	0.48173 60209	1.26817 70025	0.23254 40217
20	0.90061 29266	45 53	0.49323 01602	1.29633 07415	0.24546 30877
21	0.94564 35729	47 35	0.50333 20227	1.32575 30744	0.25847 35118
22	0.99067 42192	49 18	0.51206 72988	1.35641 09475	0.27157 41084
23	1.03570 48656	50 57	0.51949 63591	1.38829 86596	0.28476 65811
24	1.08073 55119	52 33	0.52567 71528	1.42136 28116	0.29805 07071
25	1.12576 61582	54 6	0.53066 87177	1.45557 07413	0.31142 61261
26	1.17079 68045	55 36	0.53453 12033	1.49091 75157	0.32489 18804
27	1.21582 74509	57 2	0.53732 81072	1.52734 17416	0.33844 64942
28	1.26085 80972	58 25	0.53911 06227	1.56481 68228	0.35208 79650
29	1.30588 87435	59 45	0.53994 70693	1.60330 56919	0.36581 37636
30	1.35091 93898	61 2	0.53989 28408	1.64276 08172	0.37962 48186
31	1.39595 00362	62 16	0.53900 33441	1.68316 91958	0.39350 85508
32	1.44098 06825	63 28	0.53733 39051	1.72446 24133	0.40746 37152
33	1.48601 13288	64 36	0.53493 64781	1.76660 65890	0.42149 07161
34	1.53104 19752	65 42	0.53186 06786	1.80958 73488	0.43558 13766
35	1.57607 26215	66 45	0.52818 40646	1.85326 90964	0.44972 90226
36	1.62110 32678	67 46	0.52386 33506	1.89769 45980	0.46392 94491
37	1.66613 39141	68 44	0.51902 88662	1.94278 55191	0.47817 45881
38	1.71116 45605	69 40	0.51369 13678	1.98849 21476	0.49245 35078
39	1.75619 52068	70 33	0.50788 86703	2.03476 33449	0.50676 66288
40	1.80122 58531	71 25	0.50165 60117	2.08151 68491	0.52109 87757
41	1.84625 64995	72 14	0.49502 63487	2.12878 91642	0.53544 20904
42	1.89128 71458	73 2	0.48803 01242	2.17643 86494	0.54978 90155
43	1.93631 77921	73 47	0.48069 69176	2.22443 09464	0.56412 98491
44	1.98134 84385	74 31	0.47305 24550	2.27271 60245	0.57846 24268
45	2.02637 90848	75 12	0.46512 17641	2.32133 72849	0.59274 69307
00 r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.205488385558067$, $\Theta 0 = 0.4242361430$, $HK = 1.6043008048$

B(r)	C(r)	G(r)	ψ		Fψ	90−r
1.00000 00000	3.78623 65254	0.00000 00000	90°	0′	4.05275 81695	90
0.99974 70064	3.78529 90318	0.01098 79345	89	49	4.00772 75232	89
0.99899 11477	3.78249 16163	0.02197 49829	89	38	3.96269 68769	88
0.99773 14382	3.77781 59714	0.03296 02520	89	28	3.91766 62306	87
0.99597 03726	3.77128 03065	0.04394 28343	89	17	3.87263 55842	86
0.99371 04703	3.76289 48312	0.05492 18007	89	6	3.82760 49379	85
0.99095 49588	3.75267 26317	0.06589 61931	88	54	3.78257 42916	84
0.98770 77052	3.74062 96405	0.07686 50165	88	43	3.73754 36452	83
0.98397 35058	3.72678 40000	0.08782 72314	88	32	3.69251 29989	82
0.97975 74732	3.71115 90191	0.09878 17452	88	20	3.64748 23526	81
0.97506 56227	3.69377 71248	0.10972 74034	88	8	3.60245 17063	80
0.96990 45558	3.67466 58061	0.12066 29807	87	56	3.55742 10599	79
0.96428 15032	3.65385 45535	0.13158 71709	87	44	3.51239 04136	78
0.95820 43054	3.63137 53026	0.14249 85767	87	32	3.46735 97673	77
0.95168 13014	3.60726 28114	0.15339 56986	87	19	3.42232 91209	76
0.94472 17573	3.58155 36840	0.16427 69227	87	5	3.37729 84746	75
0.93733 49419	3.55428 71880	0.17514 05085	86	52	3.33226 78283	74
0.92953 10017	3.52550 47184	0.18598 45746	86	38	3.28723 71820	73
0.92132 04850	3.49524 97967	0.19680 70842	86	24	3.24220 65356	72
0.91271 44039	3.46356 70762	0.20760 58292	86	9	3.19717 58893	71
0.90372 42062	3.43050 67437	0.21837 84126	85	54	3.15214 52430	70
0.89436 17453	3.39611 54178	0.22912 22300	85	38	3.10711 45967	69
0.88463 02503	3.36044 50445	0.23983 44495	85	22	3.06208 39503	68
0.87456 02037	3.32354 82806	0.25051 19896	85	5	3.01705 33040	67
0.86410 47610	3.28547 93300	0.26115 14957	84	48	2.97202 26577	66
0.85344 88167	3.24629 37417	0.27174 93142	84	30	2.92699 20113	65
0.84240 48716	3.20604 83874	0.28230 14649	84	11	2.88196 13650	64
0.83107 65400	3.16480 13024	0.29280 36106	83	52	2.83693 07187	63
0.81946 76845	3.12261 15798	0.30325 10250	83	32	2.79190 00724	62
0.80759 21336	3.07953 92551	0.31363 85568	83	11	2.74686 94260	61
0.79546 40460	3.03564 51012	0.32396 05923	82	49	2.70183 87797	60
0.78309 75207	2.99099 09630	0.33421 10135	82	26	2.65680 81334	59
0.77050 67624	2.94563 87432	0.34438 31544	82	3	2.61177 74870	58
0.75770 50335	2.89965 11884	0.35446 97527	81	39	2.56674 68407	57
0.74470 92077	2.85309 13269	0.36446 28984	81	13	2.52171 61944	56
0.73153 00027	2.80602 24483	0.37435 39786	80	47	2.47668 55480	55
0.71818 44065	2.75850 70940	0.38413 36176	80	19	2.43165 49017	54
0.70468 42455	2.71061 14508	0.39379 16142	79	50	2.38662 42554	53
0.69104 40537	2.66239 62465	0.40331 68729	79	20	2.34159 36091	52
0.67727 70014	2.61392 50481	0.41269 73321	78	49	2.29656 29627	51
0.66339 70061	2.56526 26633	0.42191 98869	78	17	2.25153 23164	50
0.64941 68048	2.51646 99416	0.43097 03076	77	43	2.20650 16701	49
0.63534 93209	2.46760 96971	0.43983 31542	77	8	2.16147 10238	48
0.62120 70078	2.41874 35896	0.44849 16855	76	31	2.11644 03774	47
0.60700 23531	2.36993 26700	0.45692 77651	75	52	2.07140 97311	46
0.59274 60597	2.32123 72832	0.46512 17631	75	12	2.02637 90848	45
A(r)	D(r)	E(r)	φ		Fφ	r

K = 4.3386539760, K′ = 1.5718730105, E = 1.0052585872, E′ = 1.5607201504,

r	Fφ	φ	E(r)	D(r)	Λ(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.04820 72664	2 46	0.03700 05198	1.00089 20034	0.01103 07158
2	0.09641 45328	5 31	0.07377 86240	1.00357 01005	0.02206 73080
3	0.14462 17992	8 15	0.11011 59944	1.00803 06141	0.03312 06200
4	0.19282 90656	10 59	0.14580 23384	1.01427 00982	0.04419 74541
5	0.24103 63320	13 41	0.18063 90230	1.02228 70707	0.05530 54803
6	0.28924 35984	16 21	0.21444 22668	1.03207 33471	0.06645 23081
7	0.33745 08648	18 59	0.24704 57854	1.04362 30003	0.07764 53371
8	0.38565 81312	21 34	0.27830 28485	1.05692 83230	0.08889 18240
9	0.43386 53976	24 7	0.30808 70832	1.07197 97531	0.10019 88085
10	0.48207 26640	26 37	0.33629 62300	1.08876 68632	0.11157 30046
11	0.53027 99304	29 3	0.36284 63422	1.10727 75053	0.12302 12218
12	0.57848 71968	31 27	0.38767 73064	1.12749 87762	0.13454 91483
13	0.62669 44632	33 46	0.41074 90335	1.14941 57000	0.14616 30738
14	0.67490 17296	36 2	0.43204 07437	1.17301 25520	0.15786 05139
15	0.72310 89960	38 14	0.45154 03887	1.19827 15501	0.16967 21746
16	0.77131 62624	40 23	0.46928 78531	1.22517 38302	0.18157 64770
17	0.81952 35288	42 27	0.48528 30289	1.25369 88087	0.19358 68472
18	0.86773 07952	44 28	0.49957 38319	1.28381 47103	0.20570 71870
19	0.91593 80616	46 24	0.51220 02565	1.31552 71948	0.21794 10587
20	0.96414 53280	48 16	0.52324 64512	1.34878 26000	0.23029 14612
21	1.01235 25944	50 5	0.53274 80686	1.38356 26077	0.24276 09111
22	1.06055 98608	51 50	0.54078 59033	1.41983 91820	0.25535 14914
23	1.10876 71272	53 30	0.54742 69034	1.45758 20024	0.26806 43904
24	1.15697 43936	55 7	0.55274 65730	1.49675 91731	0.28090 08608
25	1.20518 16600	56 40	0.55681 03566	1.53733 69175	0.29386 09182
26	1.25338 89264	58 10	0.55971 05941	1.57927 96919	0.30694 48879
27	1.30159 61928	59 36	0.56152 09057	1.62255 01370	0.32015 08913
28	1.34980 34592	60 58	0.56229 24153	1.66710 99581	0.33347 84147
29	1.39801 07256	62 17	0.56210 61265	1.71291 53925	0.34692 51057
30	1.44621 79920	63 33	0.56102 79658	1.75992 63260	0.36048 82928
31	1.49442 52584	64 40	0.55912 18020	1.80809 62519	0.37416 40804
32	1.54263 25248	65 55	0.55644 87917	1.85738 02809	0.38795 03444
33	1.59083 97912	67 2	0.55306 03501	1.90772 83406	0.40184 07305
34	1.63904 70576	68 6	0.54902 89975	1.95909 01488	0.41583 06538
35	1.68725 43240	69 7	0.54438 78664	2.01141 40667	0.42991 32095
36	1.73546 15904	70 5	0.53919 08711	2.06464 40451	0.44408 52267
37	1.78366 88568	71 1	0.53348 47530	2.11872 69773	0.45833 04730
38	1.83187 61232	71 54	0.52730 31847	2.17360 28173	0.47265 06209
39	1.88008 33896	72 45	0.52069 68791	2.22921 25107	0.48704 86625
40	1.92829 06560	73 34	0.51369 37297	2.28549 46808	0.50149 13298
41	1.97649 79224	74 20	0.50632 89466	2.34238 61220	0.51598 05665
42	2.02470 51888	75 5	0.49863 32034	2.39982 21493	0.53050 57822
43	2.07291 24552	75 47	0.49063 47860	2.45773 63538	0.54504 60878
44	2.12111 97216	76 58	0.48235 97411	2.51606 08149	0.55962 06569
45	2.16932 69880	77 7	0.47383 20219	2.57472 61303	0.57418 77481

$q = 0.3204003371134807$, $\quad \Theta 0 = 0.3802048484$, $\quad HK = 1.6608093153$

B(r)	C(r)	G(r)	ψ	Fψ	90−r
1.00000 00000	4.37119 23556	0.00000 00000	90° 0′	4.33865 39760	90
0.99973 05085	4.37002 95871	0.01103 73956	89 51	4.29044 67096	89
0.99892 30540	4.36654 32014	0.02207 41777	89 43	4.24223 94432	88
0.99757 07049	4.36073 80539	0.03310 97273	89 34	4.19403 21768	87
0.99570 13428	4.35262 64203	0.04414 34137	89 25	4.14582 49104	86
0.99329 11666	4.34221 80731	0.05517 45893	89 16	4.09761 76440	85
0.99035 30638	4.32953 37471	0.06620 25830	89 7	4.04941 03776	84
0.98689 15704	4.31459 15072	0.07722 66944	88 58	4.00120 31112	83
0.98291 20478	4.29741 70454	0.08824 61873	88 49	3.95299 58448	82
0.97842 03999	4.27803 82196	0.09926 02826	88 39	3.90478 85784	81
0.97342 41587	4.25648 67836	0.11026 81515	88 30	3.85658 13120	80
0.96794 03503	4.23279 78580	0.12126 89076	88 20	3.80837 40456	79
0.96193 75520	4.20700 99336	0.13226 15989	88 10	3.76016 67792	78
0.95543 48441	4.17916 47765	0.14324 51989	88 0	3.71195 95128	77
0.94855 10406	4.14930 73254	0.15421 85972	87 49	3.66375 22464	76
0.94115 92676	4.11748 55826	0.16518 05896	87 38	3.61554 49800	75
0.93331 78468	4.08375 04971	0.17612 98666	87 27	3.56733 77136	74
0.92503 92350	4.04815 58447	0.18706 50017	87 16	3.51913 04472	73
0.91633 50463	4.01075 86691	0.19798 44386	87 4	3.47092 31808	72
0.90721 07599	3.97161 62682	0.20888 64763	86 51	3.42271 59144	71
0.89770 47288	3.93079 18356	0.21976 92546	86 38	3.37450 86480	70
0.88780 42140	3.88834 85274	0.23063 07363	86 25	3.32630 13816	69
0.87753 22500	3.84435 22135	0.24146 86896	86 11	3.27809 41152	68
0.86690 32071	3.79887 07472	0.25228 06673	85 57	3.22988 68488	67
0.85594 21040	3.75197 38124	0.26306 39853	85 42	3.18167 95824	66
0.84464 32590	3.70373 27678	0.27381 56082	85 27	3.13347 23160	65
0.83302 30055	3.65422 01910	0.28453 25731	85 11	3.08526 50496	64
0.82111 72113	3.60351 12193	0.29521 10610	84 54	3.03705 77832	63
0.80893 01251	3.55168 03015	0.30584 72655	84 37	2.98885 05168	62
0.79647 89510	3.49880 44891	0.31643 69081	84 19	2.94064 32504	61
0.78377 97010	3.44496 08773	0.32697 52911	84 0	2.89243 59840	60
0.77084 61787	3.39022 76381	0.33745 72566	83 40	2.84422 87176	59
0.75769 71097	3.33468 34641	0.34787 71421	83 19	2.79602 14512	58
0.74434 72084	3.27840 74042	0.35822 87319	82 57	2.74781 41848	57
0.73081 30218	3.22147 88118	0.36850 52042	82 35	2.69960 69184	56
0.71711 40652	3.16397 71463	0.37869 90740	82 11	2.65139 96520	55
0.70325 65393	3.10598 18371	0.38880 21304	81 47	2.60319 23856	54
0.68926 59146	3.04757 21420	0.39880 53693	81 21	2.55498 51192	53
0.67514 11987	2.98882 76890	0.40869 89202	80 54	2.50677 78528	52
0.66091 11262	2.92982 49135	0.41847 19672	80 26	2.45857 05864	51
0.64654 92275	2.87064 38708	0.42811 26638	79 56	2.41036 33200	50
0.63223 14958	2.81136 10542	0.43760 80415	79 25	2.36215 60536	49
0.61778 92808	2.75205 28048	0.44694 39111	78 53	2.31394 87872	48
0.60328 92181	2.69279 48095	0.45610 47583	78 19	2.26574 15208	47
0.58873 83110	2.63366 18364	0.46507 36311	77 44	2.21753 42544	46
0.57418 77481	2.57472 64303	0.47383 20219	77 7	2.16932 69880	45
A(r)	D(r)	E(r)	φ	Fφ	r

K = 4.7427172653, K′ = 1.5712749524, E = 1.0025840855, E′ = 1.5703170190,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.05269 68585	3 1	0.04150 83698	1.00109 49202	0.00984 61866
2	0.10539 37170	6 2	0.08272 60360	1.00437 91710	0.01970 23988
3	0.15809 05755	9 1	0.12336 86879	1.00985 12249	0.02957 86287
4	0.21078 74340	11 59	0.16316 44916	1.01750 85180	0.03948 48012
5	0.26348 42925	14 56	0.20185 96235	1.02734 74434	0.04943 07415
6	0.31618 11510	17 49	0.23922 29917	1.03936 33238	0.05942 61408
7	0.36887 80095	20 40	0.27504 99964	1.05355 03843	0.06948 05245
8	0.42157 48680	23 28	0.30916 52198	1.06990 17180	0.07960 32187
9	0.47427 17265	26 13	0.34142 40166	1.08840 92458	0.08980 33181
10	0.52696 85850	28 53	0.37171 30376	1.10906 36709	0.10008 96542
11	0.57966 54435	31 30	0.39994 97772	1.13185 44382	0.11047 07636
12	0.63236 23020	34 2	0.42608 13751	1.15676 96284	0.12095 48573
13	0.68505 91605	36 30	0.45008 21300	1.18379 50985	0.13154 97806
14	0.73775 60190	38 53	0.47195 19964	1.21291 88175	0.14226 30292
15	0.79045 28775	41 12	0.49171 27333	1.24412 18480	0.15310 16203
16	0.84314 97360	43 26	0.50940 53025	1.27738 72698	0.16407 21997
17	0.89584 65946	45 35	0.52508 60758	1.31269 55075	0.17518 08788
18	0.94854 34531	47 40	0.53882 77072	1.35002 86142	0.18643 33074
19	1.00124 03116	49 40	0.55070 78595	1.38935 42806	0.19783 46027
20	1.05393 71701	51 34	0.56081 52531	1.43065 67027	0.20938 93338
21	1.10663 40286	53 25	0.56924 28378	1.47390 59633	0.22110 14976
22	1.15933 08871	55 11	0.57608 65021	1.51907 31437	0.23297 44971
23	1.21202 77456	56 52	0.58144 37172	1.56612 71505	0.24501 11103
24	1.26472 46041	58 29	0.58541 11188	1.61503 47485	0.25721 35159
25	1.31742 14626	60 2	0.58808 41618	1.66576 03865	0.26958 31846
26	1.37011 83211	61 31	0.58955 56773	1.71826 61750	0.28212 09517
27	1.42281 51796	62 55	0.58991 51945	1.77251 18082	0.29482 69565
28	1.47551 20381	64 16	0.58924 83721	1.82848 44989	0.30770 06377
29	1.52820 88966	65 33	0.58763 66017	1.88604 89185	0.32074 07202
30	1.58090 57551	66 46	0.58515 67551	1.94524 71416	0.33394 52050
31	1.63360 26136	67 56	0.58188 10541	2.00599 85969	0.34731 13599
32	1.68629 94721	69 3	0.57787 70364	2.06825 00238	0.36083 57125
33	1.73899 63306	70 6	0.57320 76019	2.13194 54360	0.37451 40449
34	1.79169 31891	71 7	0.56793 11188	2.19702 66925	0.38834 13902
35	1.84439 00476	72 4	0.56210 15757	2.26343 04764	0.40231 20314
36	1.89708 69061	72 59	0.55576 87678	2.33109 42822	0.41641 95021
37	1.94978 37646	73 51	0.54897 85058	2.39995 04116	0.43065 68890
38	2.00248 06231	74 41	0.54177 28388	2.46992 89791	0.44501 53371
39	2.05517 74816	75 28	0.53419 02851	2.54095 73266	0.45948 70563
40	2.10787 43401	76 12	0.52626 60647	2.61296 00482	0.47406 23311
41	2.16057 11986	76 55	0.51803 23206	2.68585 90255	0.48873 10316
42	2.21326 80571	77 35	0.50951 83887	2.75957 34731	0.50348 23272
43	2.26596 49156	78 14	0.50075 09241	2.83401 99984	0.51830 47025
44	2.31866 17741	78 50	0.49175 41985	2.90911 26530	0.53318 59750
45	2.37135 86326	79 25	0.48255 02516	2.98476 30422	0.54811 33155
90−r	Fψ	ψ	G(r)	C(r)	B(r)

TABLE $\theta = 88°$

$q = 0.353165648206037$, $\Theta 0 = 0.3246110213$, $HK = 1.7370861537$

B(r)	C(r)	G(r)	ψ		Fψ	90-r
1.00000 00000	5.35291 58734	0.00000 00000	90°	0′	4.74271 72653	90
0.99970 65254	5.35135 39870	0.01107 55804	89	54	4.69002 04068	89
0.99882 66090	5.34667 11120	0.02215 08037	89	47	4.63732 35483	88
0.99736 17711	5.33887 55928	0.03322 53090	89	41	4.58462 66898	87
0.99531 45401	5.32798 13106	0.04429 87274	89	35	4.53192 98313	86
0.99268 84456	5.31400 76445	0.05537 06778	89	28	4.47923 29728	85
0.98948 80069	5.29697 94165	0.06644 07630	89	21	4.42653 61143	84
0.98571 87199	5.27692 68222	0.07750 85650	89	15	4.37383 92558	83
0.98138 70401	5.25388 53459	0.08857 36405	89	8	4.32114 23973	82
0.97650 03636	5.22789 56618	0.09963 55161	89	1	4.26844 55388	81
0.97106 70046	5.19900 35203	0.11069 36828	88	54	4.21574 86803	80
0.96509 61704	5.16725 96214	0.12174 75905	88	46	4.16305 18218	79
0.95859 79343	5.13271 94744	0.13279 66420	88	39	4.11035 49633	78
0.95158 32050	5.09544 32457	0.14384 01862	88	31	4.05765 81048	77
0.94406 36948	5.05549 55939	0.15487 75112	88	23	4.00496 12463	76
0.93605 18846	5.01294 54947	0.16590 78361	88	15	3.95226 43878	75
0.92756 09875	4.96786 60538	0.17693 03026	88	6	3.89956 75293	74
0.91860 49094	4.92033 43119	0.18794 39654	87	58	3.84687 06707	73
0.90919 82095	4.87043 10392	0.19894 77822	87	48	3.79417 38122	72
0.89935 60570	4.81824 05226	0.20994 06015	87	39	3.74147 69537	71
0.88909 41880	4.76385 03454	0.22092 11507	87	29	3.68878 00952	70
0.87842 88604	4.70735 11607	0.23188 80216	87	18	3.63608 32367	69
0.86737 68071	4.64883 64589	0.24283 96552	87	8	3.58338 63782	68
0.85595 51894	4.58840 23314	0.25377 43247	86	56	3.53068 95197	67
0.84418 15481	4.52614 72300	0.26469 01166	86	45	3.47799 26612	66
0.83207 37552	4.46217 17234	0.27558 49098	86	32	3.42529 58027	65
0.81964 99644	4.39657 82526	0.28645 63526	86	19	3.37259 89442	64
0.80692 85610	4.32947 08849	0.29730 18370	86	6	3.31990 20857	63
0.79392 81128	4.26095 50677	0.30811 84711	85	52	3.26720 52272	62
0.78066 73195	4.19113 73836	0.31890 30470	85	37	3.21450 83687	61
0.76716 49636	4.12012 53075	0.32965 20072	85	21	3.16181 15102	60
0.75343 98604	4.04802 69653	0.34036 14062	85	5	3.10911 46517	59
0.73951 08099	3.97495 08972	0.35102 68681	84	48	3.05641 77932	58
0.72539 65478	3.90100 58247	0.36164 35409	84	29	3.00372 09347	57
0.71111 56987	3.82630 04227	0.37220 60448	84	10	2.95102 40762	56
0.69668 67291	3.75094 30973	0.38270 84160	83	51	2.89832 72177	55
0.68212 79036	3.67504 17706	0.39314 40446	83	30	2.84563 03592	54
0.66745 72351	3.59870 36716	0.40350 56060	83	8	2.79293 35007	53
0.65269 24519	3.52203 51359	0.41378 49862	82	44	2.74023 66422	52
0.63785 09470	3.44514 14133	0.42397 31992	82	20	2.68753 97837	51
0.62294 97425	3.36812 64840	0.43406 02965	81	55	2.63484 29252	50
0.60800 54504	3.29109 28843	0.44403 52686	81	28	2.58214 60667	49
0.59303 42368	3.21414 15421	0.45388 59368	80	59	2.52944 92081	48
0.57805 17864	3.13737 16225	0.46359 88357	80	29	2.47675 23496	47
0.56307 32704	3.06088 03834	0.47315 90851	79	58	2.42405 54911	46
0.54811 33155	2.98476 30422	0.48255 02516	79	25	2.37135 86326	45
A(r)	D(r)	E(r)	φ		Fφ	r

K = 5.4349098296, K′ = 1.5709159581, E = 1.0007616777, E′ = 1.5706767091,

r	Fφ	φ	E(r)	D(r)	A(r)
0	0.00000 00000	0° 0′	0.00000 00000	1.00000 00000	0.00000 00000
1	0.06038 78870	3 27	0.04910 51488	1.00148 76006	0.00707 98676
2	0.12077 57740	6 54	0.09795 31901	1.00595 04088	0.01507 27570
3	0.18116 36610	10 19	0.14584 05083	1.01338 83449	0.02309 10544
4	0.24155 15480	13 42	0.19348 42494	1.02380 12862	0.03204 94700
5	0.30193 94350	17 3	0.23749 17959	1.03718 89963	0.04015 09322
6	0.36232 73220	20 19	0.28055 06550	1.05358 10766	0.04833 29925
7	0.42271 52090	23 32	0.32138 60670	1.07288 62648	0.05658 38508
8	0.48310 30960	26 40	0.35977 96610	1.09519 55002	0.06492 38869
9	0.54349 09830	29 43	0.39556 46136	1.12047 55228	0.07336 81472
10	0.60387 88700	32 40	0.42862 75917	1.14872 50507	0.08191 93704
11	0.66426 67569	35 32	0.45890 52450	1.17991 15472	0.09059 80283
12	0.72465 46439	38 18	0.48637 98590	1.21412 16208	0.09941 21860
13	0.78504 25309	40 58	0.51107 40138	1.25126 06628	0.10837 25014
14	0.84543 04179	43 32	0.53304 46717	1.29138 41991	0.11748 94454
15	0.90581 83049	45 59	0.55237 70723	1.33439 44250	0.12677 26784
16	0.96620 61919	48 20	0.56917 87460	1.38042 36227	0.13623 16162
17	1.02659 40789	50 35	0.58357 38857	1.42948 15644	0.14587 50078
18	1.08698 19659	52 44	0.59569 84320	1.48160 74184	0.15571 14129
19	1.14736 98529	54 47	0.60569 45851	1.53684 68384	0.16574 82707
20	1.20775 77399	56 43	0.61370 80715	1.59348 40865	0.17599 27682
21	1.26814 56269	58 35	0.61988 74725	1.65493 85266	0.18645 13604
22	1.32853 35139	60 20	0.62437 36797	1.71727 18818	0.19712 98307
23	1.38892 14009	62 0	0.62730 67243	1.78442 86814	0.20803 33624
24	1.44930 92879	63 35	0.62881 98144	1.85248 09926	0.21916 60113
25	1.50969 71749	65 5	0.62903 92180	1.92400 88022	0.23054 16788
26	1.57008 50619	66 30	0.62808 35087	1.99881 78042	0.24214 40872
27	1.63047 29489	67 51	0.62606 38745	2.07808 98105	0.25397 22556
28	1.69086 08359	69 7	0.62308 18462	2.15848 28670	0.26605 03772
29	1.75124 87229	70 19	0.61923 26878	2.24788 01807	0.27836 77080
30	1.81163 66099	71 27	0.61460 38640	2.32284 23264	0.29092 40817
31	1.87202 44969	72 31	0.60927 36149	2.41028 34038	0.30371 78832
32	1.93241 23839	73 32	0.60331 46478	2.50017 34479	0.31674 62424
33	1.99280 02709	74 29	0.59679 24144	2.59234 80188	0.33001 67056
34	2.05318 81579	75 23	0.58976 62643	2.68700 74081	0.34349 50157
35	2.11357 60449	76 14	0.58228 07341	2.78380 63098	0.35720 59222
36	2.17396 39318	77 2	0.57441 10737	2.88275 80068	0.37113 34784
37	2.23435 18188	77 48	0.56617 36598	2.98376 78796	0.38527 07211
38	2.29473 97058	78 31	0.55761 64315	3.08678 40318	0.39960 07596
39	2.35512 75928	79 11	0.54877 42910	3.19161 70942	0.41414 17461
40	2.41551 54798	79 49	0.53967 84809	3.29835 81032	0.42888 68046
41	2.47590 33668	80 25	0.53035 60362	3.40686 60346	0.44374 44843
42	2.53629 12538	80 58	0.52083 46089	3.51642 14148	0.45879 28604
43	2.59667 91408	81 30	0.51113 37664	3.62771 82528	0.47398 04906
44	2.65706 70278	82 0	0.50127 42046	3.74032 76441	0.48932 08915
45	2.71745 49148	82 28	0.49127 37968	3.85412 04436	0.50477 27366
90-r	Fψ	ψ	G(r)	C(r)	B(r)

$q = 0.403309306338378$, $\Theta 0 = 0.2467332317$, $HK = 1.8599580878$

B(r)	C(r)	G(r)	ψ	Fψ	90-r
1.00000 00000	7.56058 07180	0.00000 00000	90° 0′	5.43490 98296	90
0.99966 43156	7.56705 29325	0.01110 10463	89 56	5.37452 19426	89
0.99865 79343	7.55944 77064	0.02220 19579	89 53	5.31413 40556	88
0.99698 28606	7.54078 94142	0.03330 25985	89 49	5.25374 61686	87
0.99464 24694	7.52910 36233	0.04440 28272	89 45	5.19335 82816	86
0.99164 14052	7.50642 60102	0.05550 24979	89 42	5.13297 03946	85
0.98798 50587	7.47880 22428	0.06660 14556	89 38	5.07258 25077	84
0.98368 21860	7.44628 78301	0.07769 95354	89 34	5.01219 46207	83
0.97874 04272	7.40894 79407	0.08879 65593	89 30	4.95180 67337	82
0.97316 92390	7.36685 71893	0.09989 23340	89 26	4.89141 88467	81
0.96697 08850	7.32009 03043	0.11098 66481	89 22	4.83103 09597	80
0.96018 44944	7.26876 73054	0.12207 92686	89 17	4.77064 30727	79
0.95279 63165	7.21296 23044	0.13316 99380	89 13	4.71025 51857	78
0.94482 96828	7.15279 40797	0.14425 83704	89 8	4.64986 72987	77
0.93629 99559	7.08838 02750	0.15534 42469	89 3	4.58947 94117	76
0.92722 44862	7.01984 61207	0.16642 72118	88 58	4.52909 15247	75
0.91761 75278	6.94732 40301	0.17750 68667	88 53	4.46870 36377	74
0.90750 02426	6.87095 31948	0.18858 27648	88 47	4.40831 57507	73
0.89689 05812	6.79087 91481	0.19965 44048	88 41	4.34792 78637	72
0.88580 82522	6.70745 33101	0.21072 12232	88 35	4.28753 99767	71
0.87427 30532	6.62023 25717	0.22178 25863	88 29	4.22715 20897	70
0.86230 78663	6.52997 87323	0.23283 77807	88 22	4.16676 42027	69
0.84993 22921	6.43665 81080	0.24388 60035	88 15	4.10637 63157	68
0.83716 91826	6.34044 09978	0.25492 63501	88 7	4.04598 84287	67
0.82404 00732	6.24150 11066	0.26595 78012	87 59	3.98560 05417	66
0.81057 05141	6.14001 55012	0.27697 92084	87 51	3.92521 26547	65
0.79678 00414	6.03610 32083	0.28798 92768	87 42	3.86482 47677	64
0.78269 56684	5.93012 56192	0.29898 65471	87 33	3.80443 68807	63
0.76833 86165	5.82208 88152	0.30996 93739	87 23	3.74404 89937	62
0.75373 17477	5.71224 68183	0.32093 59022	87 12	3.68366 11067	61
0.73890 03062	5.60074 38100	0.33188 40408	87 1	3.62327 32197	60
0.72386 73024	5.48770 09576	0.34281 14317	86 50	3.56288 53328	59
0.70865 64877	5.37358 23026	0.35371 54168	86 37	3.50249 74458	58
0.69329 05994	5.25849 10413	0.36459 29992	86 24	3.44210 95588	57
0.67779 28032	5.14260 90888	0.37544 08012	86 10	3.38172 16718	56
0.66218 59130	5.02618 66588	0.38625 50154	85 55	3.32133 37848	55
0.64649 10418	4.90773 18631	0.39703 13507	85 40	3.26094 58978	54
0.63073 30900	4.78991 03252	0.40776 49715	85 23	3.20055 80108	53
0.61493 07081	4.67189 48167	0.41845 04298	85 6	3.14017 01238	52
0.59910 86732	4.55385 49133	0.42908 15883	84 47	3.07978 22368	51
0.58327 83284	4.43595 66732	0.43965 15347	84 27	3.01939 43498	50
0.56746 83780	4.31836 23371	0.45015 24856	84 6	2.95900 64628	49
0.55169 48696	4.20123 00521	0.46057 56791	83 44	2.89861 85758	48
0.53597 61530	4.08471 30196	0.47091 12546	83 20	2.83823 06888	47
0.52032 98326	3.96896 22668	0.48114 81189	82 55	2.77784 28018	46
0.50477 27366	3.85412 04436	0.49127 37968	82 28	2.71745 49148	45
A(r)	D(r)	E(r)	φ	Fφ	r

INDEX

The numbers refer to pages.

A

B

C

D

E

PAGE

F

G

H

I

J

K

L

M